# MODULAR SHADOWS
# AND THE LÉVY–MELLIN $\infty$–ADIC TRANSFORM

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**Abstract.** This paper continues the study of the structures induced on the "invisible boundary" of the modular tower and extends some results of [MaMar1]. We start with a systematic formalism of pseudo–measures generalizing the well–known theory of modular symbols for $SL(2)$. These pseudo–measures, and the related integral formula which we call the Lévy–Mellin transform, can be considered as an "$\infty$–adic" version of Mazur's $p$–adic measures that have been introduced in the seventies in the theory of $p$–adic interpolation of the Mellin transforms of cusp forms, cf. [Ma2]. A formalism of iterated Lévy–Mellin transform in the style of [Ma3] is sketched. Finally, we discuss the invisible boundary from the perspective of non–commutative geometry.

## 0. Introduction

When the theory of modular symbols for the $SL(2)$–case had been conceived in the 70's (cf. [Ma1], [Ma2], [Sh1], [Sh2]), it was clear from the outset that it dealt with the Betti homology of some basic moduli spaces (modular curves, Kuga varieties, $\overline{M}_{1,n}$, and alike), whereas the theory of modular forms involved the de Rham and Hodge cohomology of the same spaces.

However, the combinatorial skeleton of the formalism of modular symbols is so robust, depending essentially only on the properties of continued fractions, that other interpretations and connections naturally suggest themselves.

In this paper, we develop the approach to the modular symbols which treats them as a special case of some structures supported by the "invisible boundary" of the tower of classical modular curves along the lines of [MaMar1], [MaMar2].

Naively speaking, this boundary is (the tower of) quotient space(s) of $\mathbf{P}^1(\mathbf{R})$ modulo (finite index subgroups of) $PSL(2,\mathbf{Z})$. The part of it consisting of orbits of "cusps", $\mathbf{P}^1(\mathbf{Q})$, has a nice algebraic geometric description, but irrational points are not considered in algebraic geometry, in particular, since the action of $PSL(2,\mathbf{Z})$ is highly non–discrete. This is why we call this part "invisible".

"Bad quotients" of this type can be efficiently studied using tools of Connes' non–commutative geometry. Accordingly, the Theorem 4.4.1 of [MaMar1] identified the modular homology complex with (a part of) Pimsner's exact sequence in the $K_*$–theory of the reduced crossed product algebra $C(\mathbf{P}^1(\mathbf{R})) \rtimes PSL(2,\mathbf{Z})$. Moreover, the Theorem 0.2.2 of the same paper demonstrated the existence of a version of Mellin transform (from modular forms to Dirichlet series) where the integrand was supported by the real axis rather than upper complex half–plane.

In [MaMar2], these and similar results were put in connection with the so called "holography" principle in modern theoretical physics. According to this principle, quantum field theory on a space may be faithfully reflected by an appropriate theory on the boundary of this space. When this boundary, rather than the interior, is interpreted as our observable space–time, one can proclaim that the ancient Plato's cave metaphor is resuscitated in this sophisticated guise. This metaphor motivated the title of the present paper.

Here is a review of its contents.

The sections 1–4 address the first of the two basic themes:

(i) *How does the holomorphic geometry of the upper complex half plane project itself onto the invisible boundary?*

In the first and the second sections, we introduce and develop a formalism of general pseudo–measures. They can be defined as finitely additive functions with values in an abelian group $W$ supported by the Boolean algebra generated by segments with rational ends in $\mathbf{P}^1(\mathbf{R})$. Although the definitions (and proofs) are very elementary, they capture some essential properties of the modular boundary.

In particular, in subsections 1.9–1.11 we show that the *generalized Dedekind symbols* studied by Sh. Fukuhara in [Fu1], [Fu2] are essentially certain sequences of pseudo–measures. Example 1.12 demonstrates that modular symbols are pseudo–measures. (In fact, much more general integrals along geodesics connecting cusps produce pseudo–measures; this is why we use the imagery of "projecting the holomorphic geometry of the upper complex half plane onto the invisible boundary".) Finally, subsection 2.3.1 establishes that *rational period functions* in the sense of [Kn1], [Kn2], [A], [ChZ], are pseudo–measures as well.

In the third section, we define the Lévy–Mellin transform of a pseudo–measure and prove the Theorem 3.4 generalizing the Theorem 0.2.2 of [MaMar1]. This shows that some specific Lévy functions involving pseudo–measures can serve as an efficient replacement of (non–existent) restrictions of cusp forms to the real boundary of $H$.

In the fourth section, a non–commutative version of pseudo–measures is developed. The new formalism was suggested by the theory of iterated integrals of modular forms introduced in [Ma3], [Ma4]. The iterated Lévy–Mellin transform appears naturally in this context producing some interesting multiple Dirichlet series related to but different from those discussed in [Ma3]. We hope to return to this subject later.

The second theme developed in the fifth section is:

(ii) *With what natural structure(s) is the invisible boundary endowed?*

We start with a reformulation of pseudo–measures in terms of currents on the tree $\mathcal{T}$ of $PSL(2, \mathbf{Z})$. The boundary action of $PSL(2, \mathbf{Z})$ can be best visualized in terms of the space of ends of this tree. This set of endpoints is a compact Hausdorff space which maps continuously to $\mathbf{P}^1(\mathbf{R})$ through a natural map that is $1:1$ on the irrational points and $2:1$ on the rationals.

In terms of noncommutative geometry, this boundary action is described by a crossed product $C^*$-algebra, and $W$–valued pseudo–measures can be interpreted as homomorphisms from the $K_0$ of the crossed product $C^*$-algebra to $W$.

We also introduce another $C^*$-algebra naturally associated to the boundary action, described in terms of the generalized Gauss shift of the continued fraction expansion. We show that this can be realized as a subalgebra of the crossed product algebra of the action of $PSL(2, \mathbf{Z})$ on $\partial\mathcal{T}$.

We then consider an extension of pseudo–measures to limiting pseudo–measures that parallels the notion of limiting modular symbols introduced in [MaMar1]. The limiting modular symbols can then be realized as limits of pseudo–measures associated to the ordinary modular symbols or as averages of currents on the tree of $PSL(2, \mathbf{Z})$.

## 1. Pseudo–measures: commutative case

In the following we collect the basic facts from the theory of Farey series in the form that we will use throughout the paper.

**1.1. Conventions.** We will consider $\mathbf{Q} \subset \mathbf{R} \subset \mathbf{C}$ as points of an affine line with a fixed coordinate, say, $z$. Completing this line by one point $\infty$, we get points of the projective line $\mathbf{P}^1(\mathbf{Q}) \subset \mathbf{P}^1(\mathbf{R}) \subset \mathbf{P}^1(\mathbf{C})$. *Segments* of $\mathbf{P}^1(\mathbf{R})$ are defined as non–empty connected subsets of $\mathbf{P}^1(\mathbf{R})$. The boundary of each segment generally consists of an (unordered) pair of points in $\mathbf{P}^1(\mathbf{R})$. In marginal cases, the boundary might be empty or consist of one point in which case we may speak of an *improper segment.* A proper segment is called *rational* if its ends are in $\mathbf{P}^1(\mathbf{Q})$. It is called infinite if $\infty$ is in its closure, otherwise it is called finite.

**1.2. Definition.** *A pseudo–measure on* $\mathbf{P}^1(\mathbf{R})$ *with values in a commutative group (written additively)* $W$ *is a function* $\mu : \mathbf{P}^1(\mathbf{Q}) \times \mathbf{P}^1(\mathbf{Q}) \to W$ *satisfying the following conditions: for any* $\alpha, \beta, \gamma \in \mathbf{P}^1(\mathbf{Q})$,

$$\mu(\alpha, \alpha) = 0, \quad \mu(\alpha, \beta) + \mu(\beta, \alpha) = 0\,, \tag{1.1}$$

$$\mu(\alpha, \beta) + \mu(\beta, \gamma) + \mu(\gamma, \alpha) = 0\,. \tag{1.2}$$

There are two somewhat different ways to look at $\mu$ as a version of measure.

(i) We can uniquely extend the map $(\alpha,\beta)\mapsto\mu(\alpha,\beta)$ to a finitely additive function on the Boolean algebra consisting of finite unions of *positively oriented from $\alpha$ to $\beta$* rational segments. On improper segments, in particular *points* and the whole $\mathbf{P}^1(\mathbf{R})$, this function vanishes.

Here positive orientation is defined by the increasing $z$. Thus, the ordered pair $(1,2)$ corresponds to the segment $1\le z\le 2$ (one or both ends can be excised, the pseudo–measure remains the same). However the pair $(2,1)$ corresponds to the union $\{2\le z\le\infty\}\cup\{-\infty\le z\le 1\}$ in the traditional notation (which can be somewhat misleading since $-\infty=\infty$ in $\mathbf{P}^1(\mathbf{R})$.) Thus, $(2,1)$ is an infinite segment.

(ii) Another option is to restrict oneself to finite segments and assign to $(\alpha,\beta)$ the segment with these ends *oriented from $\alpha$ to $\beta$.*

We will freely use both viewpoints, and allow ourselves some laxity about which end belongs to our segment and which not whenever it is not essential.

Moreover, working with pseudo–measures as purely combinatorial objects, we will simply identify *segments* and *ordered pairs* $(\alpha,\beta)\in\mathbf{P}^1(\mathbf{Q})$. We will call $\alpha$ (resp. $\beta$) the *ingoing* (resp. *outgoing*) end.

**1.3. The group of pseudo–measures.** $W$–valued pseudo–measures form a commutative group $M_W$ with the composition law

$$(\mu_1+\mu_2)(\alpha,\beta):=\mu_1(\alpha,\beta)+\mu_2(\alpha,\beta). \tag{1.3}$$

If $W$ is an $A$–module over a ring $A$, the pseudo–measures form an $A$–module as well.

**1.4. The universal pseudo–measure.** Let $\mathbf{Z}[\mathbf{P}^1(\mathbf{Q})]$ be a free abelian group freely generated by $\mathbf{P}^1(\mathbf{Q})$, and $\nu:\mathbf{P}^1(\mathbf{Q})\to\mathbf{Z}[\mathbf{P}^1(\mathbf{Q})]$ the tautological map. Put $\mu^U(\alpha,\beta):=\nu(\beta)-\nu(\alpha)$. Clearly, this is a pseudo–measure taking its values in the subgroup $\mathbf{Z}[\mathbf{P}^1(\mathbf{Q})]_0$, kernel of the augmentation map $\sum_i m_i\nu(\alpha_i)\mapsto\sum_i m_i$. This pseudo–measure is universal in the following sense: for any pseudo–measure $\mu$ with values in $W$, there is a homomorphism $w:\mathbf{Z}[\mathbf{P}^1(\mathbf{Q})]_0\to W$ such that $\mu=w\circ\mu^U$. In fact, we have to put $w(\nu(\beta)-\nu(\alpha))=\mu(\alpha,\beta)$.

**1.5. The action of $GL(2,\mathbf{Q})$.** The group $GL(2,\mathbf{Q})$ acts from the left upon $\mathbf{P}^1(\mathbf{R})$ by fractional linear automorphisms $z\mapsto gz$, mapping $\mathbf{P}^1(\mathbf{Q})$ to itself. It acts on pseudo–measures with values in $W$ from the right by the formula

$$(\mu g)(\alpha,\beta):=\mu(g(\alpha),g(\beta)). \tag{1.4}$$

This action is compatible with the structures described in 1.3.

The map $z \mapsto -z$ defines an involution on $M_W$, whose invariant, resp., antiinvariant points can be called even, resp. odd measures.

**1.6. Primitive segments and primitive chains.** A segment $I$ is called *primitive*, if it is rational, and if its ends are of the form $\left(\frac{a}{c}, \frac{b}{d}\right)$, $a, b, c, d \in \mathbf{Z}$ such that $ad - bc = \pm 1$. In other words, $I = (g(\infty), g(0))$ where $g \in GL(2, \mathbf{Z})$ and the group $GL(2)$ acts upon $\mathbf{P}^1$ by fractional linear transformations.

If $\det g = -1$, we can simultaneously change signs of the entries of the second column. The segment $I = (g(\infty), g(0))$ will remain the same. The intersection of the stationary subgroups of $\infty$ and 0 in $SL(2, \mathbf{Z})$ is $\pm id$. Hence the set of oriented primitive segments is a principal homogeneous space over $PSL(2, \mathbf{Z})$.

The rational ends of $I$ above are written in lowest terms, whereas $\infty$ must be written as $\frac{\pm 1}{0}$. Signs of the numerators/denominators are generally not normalized (can be inverted simultaneously), but it is natural to use $\frac{1}{0}$ for $+\infty$ and $\frac{-1}{0}$ for $-\infty$ whenever we imagine our pairs as ends of oriented segments.

Primitive segments with one infinite end are thus $(-\infty, m)$ and $(n, \infty)$, $m, n \in \mathbf{Z}$. A segment with finite ends $(\alpha, \beta)$ is primitive iff $|\alpha - \beta| = n^{-1}$ for some $n \in \mathbf{Z}$, $n \geq 1$.

Shifting a primitive segment $I = (\alpha, \beta)$ by any integer, or changing signs to $-I = (-\alpha, -\beta)$ we get a primitive segment. Hence any finite primitive segment can be shifted into $[0, 1]$, and any primitive segment $I$ of length (Lebesgue measure) $|I| \leq \frac{1}{2}$ after an appropriate shift lands entirely either in $\left[0, \frac{1}{2}\right]$, or in $\left[\frac{1}{2}, 1\right]$. Moreover, if $I = (\alpha, \beta) \subset \left[0, \frac{1}{2}\right]$ is primitive, then $1 - I =: (1 - \alpha, 1 - \beta) \subset \left[\frac{1}{2}, 1\right]$ is primitive, and vice versa.

Generally, let $\alpha, \beta \in \mathbf{P}^1(\mathbf{Q})$. Let us call *a primitive chain of length $n$ connecting $\alpha$ to $\beta$* any non–empty fully ordered family of proper primitive segments $I_1, \dots, I_n$ such that the ingoing end of $I_1$ is $\alpha$, outgoing end of $I_n$ is $\beta$, and for each $1 \leq k \leq n - 1$, the outgoing end of $I_k$ coincides with the ingoing end of $I_{k+1}$. The numeration of the segments is not a part of the structure, only their order is. We will call chains with $\alpha = \beta$ *primitive loops*, and allow one improper segment $(\alpha, \alpha)$ to be considered as a primitive loop of length 0.

This notion is covariant with respect to the $SL(2, \mathbf{Z})$–action: if $I_1, \dots, I_n$ is a primitive chain connecting $\alpha$ to $\beta$, then for any $g \in SL(2, \mathbf{Z})$, $g(I_1), \dots, g(I_n)$ is a primitive chain connecting $g(\alpha)$ to $g(\beta)$.

**1.6.1. Lemma.** *(a) If $I_1$, $I_2$ are two open primitive segments and at least one of them is finite, then either $I_1 \cap I_2 = \emptyset$, or one of them is contained in another.*

*(b) Any two points $\alpha, \beta \in \mathbf{P}^1(\mathbf{Q})$ can be connected by a primitive chain.*

**Proof.** This is well known. We reproduce an old argument showing (b) from [Ma1] in order to fix some notation. Consider the following sequence of *normalized convergents to $\alpha$*

$$\frac{p_{-1}(\alpha)}{q_{-1}(\alpha)} := \frac{1}{0} = \infty, \ \frac{p_0(\alpha)}{q_0(\alpha)} := \frac{p_0}{1}, \ \dots \ , \frac{p_n(\alpha)}{q_n(\alpha)} = \alpha \tag{1.5}$$

Here $\alpha \in \mathbf{Q}$, $p_0 := [\alpha]$ the integer part of $\alpha$, and convergents are calculated from

$$\alpha = k_0(\alpha) + [k_1(\alpha), \dots, k_n(\alpha)] := p_0 + \frac{1}{k_1(\alpha) + \frac{1}{k_2(\alpha) + \dots \frac{1}{k_n(\alpha)}}}$$

with $1 \le k_i(\alpha) \in \mathbf{Z}$ for $i \ge 1$ and $k_n(\alpha) \ge 2$ whenever $\alpha \notin \mathbf{Z}$ so that $n \ge 1$.

The sequence

$$I_k(\alpha) := \left( \frac{p_k(\alpha)}{q_k(\alpha)}, \frac{p_{k+1}(\alpha)}{q_{k+1}(\alpha)} \right) \tag{1.6}$$

is a primitive chain connecting $\infty$ to $\alpha$.

Applying this construction to $\beta$ and reversing the sequence of ends (1.5), we get a primitive chain connecting $\beta$ to $\infty$. Joining chains from $\alpha$ to $\infty$ and from $\infty$ to $\beta$, we get a chain from $\alpha$ to $\beta$.

We put also

$$g_k(\alpha) := \begin{pmatrix} p_k(\alpha) & (-1)^{k+1} p_{k+1}(\alpha) \\ q_k(\alpha) & (-1)^{k+1} q_{k+1}(\alpha) \end{pmatrix} \in SL(2, \mathbf{Z}) \tag{1.7}$$

so that

$$I_k(\alpha) = (g_k(\alpha)(\infty), g_k(\alpha)(0)) \,. \tag{1.8}$$

**1.7. Corollary.** *Each pseudo–measure is completely determined by its values on primitive segments.*

In fact, for any $\alpha$, $\beta$ and any primitive chain $I_1, \dots, I_n$ connecting $\alpha$ to $\beta$, we must have

$$\mu(\alpha, \beta) = \sum_{k=1}^{n} \mu(I_k). \tag{1.9}$$

**1.7.1. Definition.** *A pre–measure $\widetilde{\mu}$ is any $W$–valued function defined on primitive segments and satisfying relations (1.1) and (1.2) for all primitive chains of length $\leq$ 3.*

Clearly, restricting a pseudo–measure to primitive segments, we get a pre–measure. We will prove the converse statement.

**1.8. Theorem.** *Each pre–measure $\widetilde{\mu}$ can be uniquely extended to a pseudo–measure $\mu$.*

**Proof.** If such a pseudo–measure exists, it is defined by the (family of) formula(s) (1.9):

$$\mu(\alpha,\beta) := \sum_{k=1}^{n} \widetilde{\mu}(I_k). \tag{1.10}$$

We must only check that this prescription is well defined, that is, does not depend on the choice of $\{I_k\}$. Our argument below is somewhat more elaborate than what is strictly needed here. Its advantage is that it can be applied without changes to the proof of the Theorem 4.3 below, which is a non–commutative version of the Theorem 1.8.

We will first of all define four types of *elementary moves* which transform a primitive chain $\{I_k\}$ connecting $\alpha$ to $\beta$ to another such primitive chain without changing the r.h.s. of (1.10).

(i) Choose $k$ (if it exists) such that $I_k = (\gamma,\delta)$, $I_{k+1} = (\delta,\gamma)$, and delete $I_k, I_{k+1}$ from the chain.

This does not change (1.10) because $\widetilde{\mu}(\gamma,\delta) + \widetilde{\mu}(\delta,\gamma) = 0$.

(ii) A reverse move: choose a point $\gamma$ which is the outgoing end of some $I_k$ (resp. $\gamma = \alpha$), choose a primitive segment $(\gamma,\delta)$, and insert the pair $(\gamma,\delta),(\delta,\gamma)$ right after $I_k$ (resp. before $I_1$.)

This move can also be applied to the empty loop connecting $\gamma$ to $\gamma$, then it produces a chain of length two. Again, application of such a move is compatible with (1.10).

(iii) Choose $I_k, I_{k+1}, I_{k+2}$ (if they exist) such that these three segments have the form $(\gamma_1,\gamma_2),(\gamma_2,\gamma_3),(\gamma_3,\gamma_1)$, and delete them from the chain.

This move is compatible with (1.10) as well because we have postulated (1.2) on such chains.

(iv) A reverse move: choose a point $\gamma$ which is the outgoing end of some $I_k$ (resp. $\gamma = \alpha$), and insert any triple of segments as above right after $I_k$ (resp. before $I_1$.)

Now we will show that

(*) *any primitive loop, that is, a primitive chain with $\alpha = \beta$, can be transformed into an empty loop by a sequence of elementary moves.*

Suppose that we know this. If $I_1, \dots, I_n$, $J_1, \dots, J_m$ are two chains connecting $\alpha$ to $\beta$, we can produce from them a loop connecting $\alpha$ to $\alpha$ by putting after $I_n$ the segments $J_m, \dots, J_1$ with reversed orientations. The r.h.s. part of (1.10) calculated for this loop must be zero because it is zero for the empty loop. Hence $I_1, \dots, I_n$ and $J_1, \dots, J_m$ furnish the same r.h.s. of (1.10).

We will establish (*) by induction on the length of a loop. We will consider several cases separately.

*Case 1: loops of small length.* The discussion of the elementary moves above shows that loops of length 2 or 3 can be reduced to an empty loop by one elementary move. Loops of length 1 do not exist.

*Case 2: existence of a subloop.* Assume that the ingoing end of some $I_k$, $k \geq 1$, coincides with the outgoing end of some $I_l$, $k+1 \leq l \leq n-1$. Then the segments $I_k, \dots, I_l$ form a subloop of lesser length. By induction, we may assume that it can be reduced to an empty loop by a sequence of elementary moves. The same sequence of moves diminishes the length of the initial loop.

From now on, we may and will assume that our loop $I_1, \dots, I_n$ is of length $n \geq 4$ and does not contain proper subloops.

Applying an appropriate $g \in PSL(2, \mathbf{Z})$, we may and will assume that $\alpha = \beta = \infty$. Thus our loop starts with $I_1 = (\infty, a)$ and ends with $I_n = (b, \infty)$, $a, b \in \mathbf{Z}$, $a \neq b$ because of absence of subloops.

*Case 3:* $|a - b| = 1$. We will consider the case $b = a + 1$; the other one reduces to this one by the change of orientation. We may also assume that $I_2, \dots, I_{n-1} \subset [a, a+1]$, because otherwise $\infty$ would appear once again as one of the vertices.

Since $n \geq 4$, the primitive chain $I_2, \dots, I_{n-1}$ connecting $a$ to $a+1$ has length at least 2. We will apply to the initial loop the following elementary moves: (i) insert $((a, a+1), (a+1, a)$ after $I_1 = (\infty, a)$; (ii) delete $(\infty, a), ((a, a+1), (a+1, \infty)$. The resulting loop connecting $a$ to $a$ has length $n-1$ and can be reduced to the empty loop by the inductive assumption.

*Case 4:* $|a - b| \geq 2$. Again, we may assume that $a < b$ and that $I_2, \dots, I_{n-1} \subset [a, b]$. Consider two subcases.

It can be that the Lebesgue measure of $I_2$ is 1, so that $I_2 = (a, a+1)$. Then we apply two elementary moves: (i) insert $(a+1, \infty), (\infty, a+1)$ after $I_2$; (ii) delete $(\infty, a) = I_1, (a, a+1) = I_2, (a+1, \infty)$. We will get a loop of length $n-1$.

If the Lebesgue measure of $I_2$ is $< 1$, then some initial subchain $I_2, \dots, I_k$ of length $\geq 2$ will connect $a$ to $a+1$. In this subcase, we will apply the following sequence of elementary moves: (i) insert $(a, a+1), (a+1, a)$ after $I_1 = (\infty, a)$; (ii) insert $(a+1, \infty), (\infty, a+1)$ after $I_k$; (iii) delete $(\infty, a), (a, a+1), (a+1, \infty)$.

The resulting loop will have length $n+1$, however, it will also have a subloop $I_2, \dots, I_k, (a+1, a)$ of length $\leq n-1$ and $\geq 3$. The latter can be deleted by elementary moves in view of the inductive assumption leaving the loop of length $\leq n-2$.

This completes the proof of (*) and of the Theorem 1.8.

**1.9. Pseudo–measures and generalized Dedekind symbols.** Let $V := \{(p,q) \in \mathbf{Z}^2 \,|\, p \geq 1, \gcd(p,q) = 1\}$.

Slightly extending the definition given in [Fu1], [Fu2], we will call *a $W$–valued generalized Dedekind symbol* any function $D : V \to W$ satisfying the functional equation

$$D(p,q) = D(p, q+p)\,. \tag{1.11}$$

The symbol $D$ can be reconstructed (at least, in the absence of 2–torsion in $W$) from its *reciprocity function* $R$ defined on the subset $V_0 := \{(p,q) \in V \,|\, q \geq 1\}$ by the equation

$$R(p,q) := D(p,q) - D(q,-p)\,. \tag{1.12}$$

From (1.11) we get a functional equation for $R$:

$$R(p+q, q) + R(p, p+q) = R(p,q)\,. \tag{1.13}$$

Fukuhara in [Fu1] considers moreover the involution $(p,q) \mapsto (p,-q)$. If $D$ is even with respect to such an involution, its reciprocity function satisfies the additional condition $R(1,1) = 0$, which together with (1.13) suffices for existence of $D$ with reciprocity function $R$.

In the following, we will work with reciprocity functions only.

**1.9.1. From pseudo–measures to reciprocity functions.** Consider the set $\Pi$ consisting of all primitive segments contained in $[0,1]$.

If $a/p < b/q$ are ends of $I \in \Pi$ written in lowest terms with $p, q > 0$, we have $(p,q) \in V_0$. We have thus defined a map $\Pi \to V_0$. One easily sees that it is a bijection.

The involution on $\Pi$ which sends $(p,q)$ to $(q,p)$ corresponds to $I \mapsto 1 - I$.

Let $\mu$ be a $W$–valued pseudo–measure. Define a function $R_{\mu,0}$ which in the notation of the previous paragraph is given by

$$R_{\mu,0}(p,q) := \mu\left(\frac{a}{p}, \frac{b}{q}\right)\,. \tag{1.14}$$

Furthermore, for each $n \in \mathbf{Z}$ put

$$R_{\mu,n}(p,q) := \mu\left(n+\frac{a}{p}, n+\frac{b}{q}\right). \tag{1.15}$$

**1.9.2. Proposition.** *(a) Equations (1.14) and (1.15) determine reciprocity functions.*

*(b) We have $R_{\mu,n}(1,1)=0$ iff $\mu(n,n+1)=0$.*

**Proof.** In order to prove (a) it suffices to establish the equation (1.13) for $R_{\mu,n}$. When $n=0$, this follows from (1.2) applied to the Farey triple

$$\alpha=\frac{a}{p},\ \beta=\frac{a+b}{p+q},\ \gamma=\frac{b}{q}\,.$$

To get the general case, one simply shifts this triple by $n$.

Since $R_{\mu,0}(1,1)=\mu(0,1)$, we get (b).

**1.10. From reciprocity functions to pseudo–measures.** Consider now any sequence of reciprocity functions $R_n$, $n \in \mathbf{Z}$, and an element $\omega \in W$. Construct from this data a $W$–valued function $\widetilde{\mu}$ on the set of all primitive segments by the following prescriptions. For positively oriented infinite segments we put:

$$\widetilde{\mu}(-\infty,0) := \omega\,, \tag{1.16}$$

and moreover, when $n \in \mathbf{Z}$, , $n \geq 1$,

$$\widetilde{\mu}(-\infty,n) := \omega + R_0(1,1) + R_1(1,1) + \dots + R_{n-1}(1,1)\,, \tag{1.17}$$

$$\widetilde{\mu}(-\infty,-n) := \omega - R_{-1}(1,1) - \dots - R_{-n}(1,1)\,, \tag{1.18}$$

$$\widetilde{\mu}(-n,\infty) := R_{-n}(1,1) + R_{-n+1}(1,1) + \dots + R_{-1}(1,1) - \omega\,, \tag{1.19}$$

$$\widetilde{\mu}(n,\infty) := -R_{n-1}(1,1) - R_{n-2}(1,1) - \dots - R_0(1,1) - \omega\,. \tag{1.20}$$

For positively oriented finite segments we put:

$$\widetilde{\mu}(n+\alpha,n+\beta) := R_n(p,q) \tag{1.21}$$

if $0 \leq \alpha = a/p < \beta = p/q \leq 1$.

Finally, for negatively oriented primitive segments we prescribe the sign change, in concordance with (1.2):

$$\widetilde{\mu}(\beta,\alpha) = -\widetilde{\mu}(\alpha,\beta). \tag{1.22}$$

**1.11. Theorem.** *The function $\widetilde{\mu}$ is a pre–measure. Therefore it can be uniquely extended to a pseudo–measure $\mu$.*

*The initial family $\{R_n, \omega\}$ can be reconstructed from $\mu$ with the help of (1.15) and (1.16).*

**Proof.** In view of the Theorem 1.8, it remains only to check the relations (1.2) for primitive loops of length 3.

If all ends in such a loop are finite, they form a Farey triple in $[0,1]$ or a Farey triple shifted by some $n \in \mathbf{Z}$. In this case (1.2) follows from the functional equations for $R_n$.

If one end in such a loop is $\infty$, then up to an overall change of orientation it has the form

$$(-\infty, n), (n, n+1), (n+1, \infty)\,.$$

Straightforward calculations using (1.16)–(1.21) complete the proof.

**1.12. Example: pseudo–measures associated with holomorphic functions vanishing at cusps.** The spaces $\mathbf{P}^1(\mathbf{Q})$ and $\mathbf{P}^1(\mathbf{R})$ are embedded into the Riemann sphere $\mathbf{P}^1(\mathbf{C})$ using the complex values of the same affine coordinate $z$. The infinity point acquires one more traditional notation $i\infty$. The upper half plane $H = \{z \,|\, \mathrm{Im}\, z > 0\}$ is embedded into $\mathbf{P}^1(\mathbf{C})$ as an open subset with the boundary $\mathbf{P}^1(\mathbf{R})$. The metric $ds^2 := \dfrac{|dz|^2}{(\mathrm{Im}\, z)^2}$ has the constant negative curvature $-1$, and the fractional linear transformations $z \mapsto gz$, $g \in GL^+(2, \mathbf{Z})$, act upon $H$ by holomorphic isometries.

Let $\mathcal{O}(H)_{cusp}$ be the space of holomorphic functions $f$ on $H$ having the following property: *when we approach a cusp $\alpha \in \mathbf{P}^1(\mathbf{Q})$ along a geodesic leading to this cusp, $|f(z)|$ vanishes faster than $l(z_0, z)^{-N}$ for all integers $N$, where $z_0$ is any fixed reference point in $H$, and $l(z_0, z)$ is the exponential of the geodesic distance from $z_0$ to $z$.*

If $f \in \mathcal{O}(H)_{cusp}$, the map $\mathbf{P}^1(\mathbf{Q})^2 \to \mathbf{C}$:

$$(\alpha, \beta) \mapsto \int_\alpha^\beta f(z)dz \tag{1.23}$$

is well defined and satisfies (1.1), (1.2). (Here and henceforth we always tacitly assume that the integration path, at least in some vicinity of $\alpha$ and $\beta$, follows a geodesic.)

Considering the r.h.s. of (1.23) as a linear functional of $f$ i.e., an element of the linear dual space $\mathbf{W} := (\mathcal{O}(H)_{cusp})^*$, we get our basic $\mathbf{W}$–valued pseudo–measure

$\mu$. The classical constructions with modular forms introduce some additional structures and involve passage to a quotient of $\mathbf{W}$, cf. 2.6 below.

The real and imaginary parts of (1.23) are pseudo–measures as well.

More generally, one can replace the integrand in (1.23) by a closed real analytic (or even smooth) 1–form in $H$ satisfying the same exponential vanishing condition along cusp geodesics as above.

**1.13. $p$–adic analogies.** The segment $[-1,1]$ is determined by the condition $|\alpha|_\infty \le 1$ so that it traditionally is considered as an analog of the ring of $p$–adic integers $\mathbf{Z}_p$ determined by $|\alpha|_p \le 1$. Less traditionally, we suggest to consider the open primitive (Farey) segments of length $\le 1/2$ in $[-1,1]$ as analogs of residue classes $a \bmod p^m$. Both systems of subsets share the following property: any two subsets either do not intersect, or one of them is contained in another one. The number $m$ then corresponds to the Farey depth of the respective segment that is, to the length of the continued fraction decomposition of one end.

The notion of a pseudo–measure is similar to that of $p$–adic measure: see [Ma2], sec. 8 and 9. This list of analogies will be continued in the section 3.5 below.

## 2. Modular pseudo–measures.

**2.1. Modular pseudo–measures.** Let $\Gamma \subset SL(2,\mathbf{Z})$ be a subgroup of finite index. In this whole section we assume that the group of values $W$ of our pseudo–measures is a left $\Gamma$–module. The action is denoted $\omega \mapsto g\omega$ for $g \in \Gamma$, $\omega \in W$.

**2.1.1. Definition.** *A pseudo–measure $\mu$ with values in $W$ is called modular with respect to $\Gamma$ if $\mu g(\alpha,\beta) = g[\mu(\alpha,\beta)]$ for any $(\alpha,\beta)$ and $g \in \Gamma$. Here $\mu g(\alpha,\beta)$ is defined by (1.4).*

We denote by $M_W(\Gamma)$ the group of all such pseudo–measures. More generally, if $\chi$ is a character of $\Gamma$ and if multiplication by $\chi(g)$, $g \in \Gamma$, makes sense in $W$ (e. g., $W$ is a module over a ring where the values of $\chi$ lie), we denote by $M_W(\Gamma,\chi)$ the group of pseudo–measures $\mu$ such that $\mu g(\alpha,\beta) = \chi(g) \cdot g\mu(\alpha,\beta)$.

**2.1.2. Example.** Let $\Gamma = SL(2,\mathbf{Z})$. Then from the Corollary 1.7 one infers that any $\Gamma$–modular pseudo–measure $\mu$ is uniquely determined by its single value $\mu(\infty,0)$, because if $I = g(\infty,0)$ is a primitive interval, then we must have $\mu(I) = g\mu(\infty,0)$. In particular,

$$\mu(\infty,\alpha) = \sum_{k=-1}^{n-1} g_k(\alpha)\mu(\infty,0) \tag{2.1}$$

where the matrices $g_k(\alpha)$ are defined in (1.7).

For a general $\Gamma$, denote by $\{h_k\}$ a system of representatives of the coset space $\Gamma\backslash SL(2,\mathbf{Z})$. Then the similar reasoning shows that any $\Gamma$–modular pseudo–measure $\mu$ is uniquely determined by its values $\mu(h_k(\infty),h_k(0))$.

**2.1.3. Modularity and generalized Dedekind symbols.** Assume that $W$ is fixed by the total group of shifts in $SL(2,\mathbf{Z})$ fixing $\infty$: $z\mapsto z+n$, $n\in\mathbf{Z}$. Assume moreover that $\mu$ is modular with respect to some subgroup (non–necessarily of finite index) containing the total group of shifts. Then from (1.15) one sees that that all the reciprocity functions $R_{\mu,n}$ coincide with $R_\mu := R_{\mu,0}$, and from (1.17)–(1.20) it follows that $R(1,1)=0$. Hence $\mu$ is associated with a generalized Dedekind symbol. Conversely, starting with such a symbol, we can uniquely reconstruct a shift–invariant pseudo–measure.

**2.2. A description of $M_W(SL(2,\mathbf{Z}))$.** Consider now the map

$$M_W(SL(2,\mathbf{Z}))\to W:\ \mu\mapsto\mu(\infty,0)\,. \tag{2.2}$$

From 2.1.1 it follows that this is an injective group homomorphism. Its image is constrained by several conditions.

Firstly, because of modularity, we have

$$(-id)\mu(\infty,0)=\mu(\infty,0)\,. \tag{2.3}$$

Therefore, the image of (2.2) is contained in $W_+$, the subgroup of fixed points of $-id$. Hence we may and will consider it as a module over $PSL(2,\mathbf{Z})$. The latter group is the free product of its two subgroups $\mathbf{Z}_2$ and $\mathbf{Z}_3$ generated respectively by the fractional linear transformations with matrices

$$\sigma=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad \tau=\begin{pmatrix}0&-1\\1&-1\end{pmatrix}. \tag{2.4}$$

Now, $\sigma(\infty)=0$, $\sigma(0)=\infty$. Hence secondly,

$$(1+\sigma)\mu(\infty,0)=0\,. \tag{2.5}$$

And finally,

$$\tau(0)=1,\ \tau(1)=\infty,\ \tau(\infty)=0\,.$$

so that the modularity implies

$$(1+\tau+\tau^2)\mu(\infty,0)=0\,. \tag{2.6}$$

To summarize, we get the following complex

$$0 \to M_W(SL(2,\mathbf{Z})) \to W_+ \to W_+ \times W_+ \tag{2.7}$$

where the first arrow is injective, and the last arrow is $(1+\sigma, 1+\tau+\tau^2)$.

**2.3. Theorem.** *The sequence (2.7) is exact. In other words, the map* $\mu \mapsto \mu(\infty, 0)$ *induces an isomorphism*

$$M_W(SL(2,\mathbf{Z})) \cong \mathrm{Ker}\,(1+\sigma) \cap \mathrm{Ker}\,(1+\tau+\tau^2)|_{W_+}\,. \tag{2.8}$$

**Proof.** Choose an element $\omega$ in the r.h.s. of (2.8). Define a $W$–valued function $\widetilde{\mu}$ on primitive segments by the formula

$$\widetilde{\mu}(g(\infty), g(0)) := g\omega\,. \tag{2.9}$$

We will check that it is a pre–measure in the sense of 1.7.1. This will show that any such $\omega$ comes from a (unique) measure $\mu$. It is then easy to check that $\mu$ is in fact $SL(2,\mathbf{Z})$–modular: from the formula (1.9) and the independence of its r.h.s. from the choice of a primitive chain $I_k = g_k(\infty, 0)$ we get using (2.9):

$$\mu(g(\alpha), g(\beta)) = \sum_{k=1}^{n} \widetilde{\mu}(g(I_k)) = \sum_{k=1}^{n} \widetilde{\mu}(gg_k(\infty, 0)) =$$

$$\sum_{k=1}^{n} gg_k(\omega) = \sum_{k=1}^{n} g\widetilde{\mu}(I_k) = g\mu(\alpha, \beta).$$

The property (1.1) holds for $\widetilde{\mu}$ in view of (2.5). In fact, if $(\alpha, \beta) = g(\infty, 0)$, then

$$(\beta, \alpha) = g(0, \infty) = g\sigma(\infty, 0)$$

so that

$$\widetilde{\mu}(\alpha, \beta) + \widetilde{\mu}(\beta, \alpha) = g\omega + g\sigma\omega = 0.$$

Similarly, the property (1.2) follows from (2.6). In order to deduce this, we must check that both types of primitive loops of length 3 considered in the proof of the Theorem 1.8 can be represented in the form

$$g(\infty, 0),\ g\tau(\infty, 0),\ g\tau^2(\infty, 0)$$

for an appropriate $g$. We leave this as an easy exercise.

**2.3.1. Rational period functions as pseudo–measures.** Fix an integer $k \geq 0$ and denote by $W_k$ the space of higher differentials $\omega(z) := q(z)(dz)^k$ where $q(z)$ is a rational function. Define the left action of $PSL(2, \mathbf{Z})$ on $W_k$ by $(g\omega)(z) := \omega(g^{-1}(z))$. Then the right hand side of (2.8) consists of such $q(z)(dz)^k$ that $q(z)$ satisfies the equations

$$q(z) + z^{-2k} q(\frac{-1}{z}) = 0, \quad q(z) + z^{-2k} q(1 - \frac{1}{z}) + (z-1)^{-2k} q(\frac{1}{1-z}) = 0.$$

These equations define the space of *rational period functions of weight* $2k$. M. Knopp introduced them in [Kn1] and showed that such functions can have poles only at 0 and real quadratic irrationalities (the latter are "shadows" of closed geodesics on modular curves.) Y. J. Choie and D. Zagier in [ChZ] provided a very explicit description of them. Finally, A. Ash studied their generalizations for arbitrary $\Gamma$, which can be treated as pseudo–measures using the construction described in the next subsection.

**2.4. Induced pseudo–measures.** By changing the group of values $W$, we can reduce the description of $M_W(\Gamma)$ for general $\Gamma$ to the case $\Gamma = SL(2, \mathbf{Z})$.

Concretely, given $\Gamma$ and a $\Gamma$–module $W$, put

$$\widehat{W} := \mathrm{Hom}_\Gamma(PSL(2, \mathbf{Z}), W)\,. \tag{2.10}$$

An element $\varphi \in \widehat{W}$ is thus a map $g \mapsto \varphi(g) \in W$ such that $\varphi(\gamma g) = \gamma\varphi(g)$ for all $\gamma \in \Gamma$ and $g \in SL(2, \mathbf{Z})$. Such functions form a group with pointwise addition, and $PSL(2, \mathbf{Z})$ acts on it from the left via $(g\varphi)(\gamma) := \varphi(\gamma g)$.

Any $\widehat{W}$–valued $SL(2, \mathbf{Z})$–modular measure $\hat{\mu}$ induces a $W$–valued $\Gamma$–modular measure:

$$\mu(\alpha, \beta) := (\hat{\mu}(\alpha, \beta))(1_{SL(2,\mathbf{Z})})\,. \tag{2.11}$$

Conversely, any $W$–valued $\Gamma$–modular measure $\mu$ induces a $\widehat{W}$–valued $SL(2, \mathbf{Z})$–modular measure $\hat{\mu}$:

$$(\tilde{\mu}(\alpha, \beta))(g) := \mu(g(\alpha), g(\beta))\,. \tag{2.12}$$

**2.4.1. Proposition.** *The maps (2.11), (2.12) are well defined and mutually inverse. Thus, they produce a canonical isomorphism*

$$M_W(\Gamma) \cong M_{\widehat{W}}(SL(2, \mathbf{Z})). \tag{2.13}$$

The proof is straightforward.

**2.4.2. Modular pseudo–measures and cohomology.** Given $\mu \in M_W(\Gamma)$ and $\alpha \in \mathbf{P}^1(\mathbf{Q})$, consider the function

$$c^{\mu}_{\alpha} = c_\alpha : \Gamma \to W, \quad c_\alpha(g) := \mu(g\alpha, \alpha).$$

From (1.1), (1.2) and the modularity of $\mu$ it follows that this is an 1–cocycle in $Z^1(\Gamma, W)$:

$$c_\alpha(gh) = c_\alpha(g) + gc_\alpha(h)\,.$$

Changing $\alpha$, we get a cohomologous cocycle:

$$c_\alpha(g) - c_\beta(g) = g\mu(\alpha,\beta) - \mu(\alpha,\beta)\,.$$

If we restrict $c_\alpha$ upon the subgroup $\Gamma_\alpha$ fixing $\alpha$, we get the trivial cocycle, so the respective cohomology class vanishes. Call a cohomology class in $H^1(\Gamma, W)$ *cuspidal* if it vanishes after restriction on each $\Gamma_\alpha$.

Thus, we have a canonical map of $\Gamma$–modular pseudo–measures to the cuspidal cohomology

$$M_W(\Gamma) \to H^1(\Gamma, W)_{cusp}\,.$$

If $\Gamma = PSL(2, \mathbf{Z})$, both groups have more compact descriptions. In particular, the map

$$Z^1(PSL(2,\mathbf{Z}), W_+) \mapsto \mathrm{Ker}\,(1+\sigma) \times \mathrm{Ker}\,(1+\tau+\tau^2) \subset W_+ \times W_+ :\ c \mapsto (c(\sigma), c(\tau))$$

is a bijection.

The value $\mu(\infty, 0) = -c_\infty(\sigma)$ furnishes the connection between the two descriptions which takes the following form:

**2.4.3. Proposition.** *(i) For any $PSL(2,\mathbf{Z})$–modular pseudo–measure $\mu$, we have*

$$c^{\mu}_{\infty}(\sigma) = c^{\mu}_{\infty}(\tau) = -\mu(\infty, 0) \in W_+\,.$$

*(ii) This correspondence defines a bijection*

$$M_W(PSL(2,\mathbf{Z})) \cong Z^1(PSL(2,\mathbf{Z}), W_+)_{cusp}$$

*where the latter group by definition consists of cocycles with equal components $c(\sigma) = c(\tau)$.*

In the sec. 4.8.2, this picture will be generalized to the non–commutative case.

**2.5. Action of the Hecke operators on pseudo–measures.** To define the action of the Hecke operators upon $M_W(\Gamma)$, we have to assume that the group of values $W$ is a left module not only over $\Gamma$ (as up to now) but in fact over $GL^+(2,\mathbf{Q})$. We adopt this assumption till the end of this section.

Then $M_W(\Gamma)$ becomes a $GL^+(2,\mathbf{Q})$–bimodule, with the right action (1.4) and the left one

$$(g\mu)(\alpha,\beta) := g[\mu(\alpha,\beta)]\,. \tag{2.14}$$

Let $\Delta$ be a double coset in $\Gamma \setminus GL^+(2,\mathbf{Q})/\Gamma$, or a finite union of such cosets. Denote by $\{\delta_i\}$ a complete finite family of representatives of $\Gamma\setminus\Delta$.

**2.5.2. Proposition.** *The map*

$$T_\Delta:\ \mu\mapsto\mu_\Delta := \sum_i \delta_i^{-1}\mu\delta_i \tag{2.15}$$

*restricted to $M_W(\Gamma)$ depends only on $\Delta$ and sends $M_W(\Gamma)$ to itself.*

**Proof.** If $\{\delta_i\}$ are replaced by $\{g_i\delta_i\}$, $g_i\in\Gamma$, the r.h.s. of (2.15) gets replaced by

$$\sum_i \delta_i^{-1}g_i^{-1}\mu g_i\delta_i\,.$$

But $g_i^{-1}\mu g_i=\mu$ on $M_W(\Gamma)$.

To check that $\mu_\Delta g = g\mu_\Delta$ for $g\in\Gamma$, notice that the right action of $\Gamma$ induces a permutation of the set $\Gamma\setminus\Delta$ so that for each $g\in\Gamma$,

$$\delta_i g = g'(i,g)\cdot\delta_{j(i,g)},\qquad g'(i,g)\in\Gamma,$$

and $i\mapsto j(i,g)$ is a permutation of indices.

Hence

$$\delta_i^{-1}\mu\delta_i g = \delta_i^{-1}\mu g'(i,g)\cdot\delta_{j(i,g)} = \delta_i^{-1}g'(i,g)\mu\cdot\delta_{j(i,g)}.$$

But $\delta_i^{-1}g'(i,g) = g\delta_{j(i,g)}^{-1}$. Therefore,

$$\delta_i^{-1}\mu\delta_i g = g\delta_{j(i,g)}^{-1}\mu\delta_{j(i,g)},$$

and while $g$ is kept fixed, the summation over $i$ produces the same result as summation over $j(i,g)$. This completes the proof.

Notice in conclusion that all elements of one double coset $\Delta$ have the same determinant, say, $D$. Usually one normalizes Hecke operators by choosing $\Delta\subset M(2,\mathbf{Z})$ and replacing $T_\Delta$ by $DT_\Delta$.

**2.5.3. Classical Hecke operators $T_n$.** They correspond to the case $\Gamma = SL(2,\mathbf{Z})$. Let $\Delta_n$ be the finite union of the double classes contained in $M_n(2,\mathbf{Z})$:

matrices with integer entries and determinant $n$. It is well known that the following matrices form a complete system of representatives of $\Gamma \setminus \Delta_n$:

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad ad = n,\ 1 \le b \le d\,. \tag{2.16}$$

We put $T_n := T_{\Delta_n}$.

**2.6. Pseudo–measures associated with cusp forms.** The classical theory of modular symbols associated with cusp forms of arbitrary weight with respect to a group $\Gamma$ furnishes the basic examples of modular pseudo–measures. They are specializations of the general construction of sec. 1.12.

We will recall the main formulas and notation.

First of all, one easily sees that the space $\mathcal{O}(H)_{cusp}$ is a $\mathbf{C}[z]$–module, and simultaneously a right $GL^+(2,\mathbf{Z})$–module with respect to the "variable change" action. The change from the right to the left comes as a result of dualization, cf. the last paragraph of sec. 1.

In the classical theory, one restricts the pseudo–measure (1.23) onto subspaces of $\mathcal{O}(H)_{cusp}$ consisting of the products $f(z)P(z)$ where $f$ is a classical cusp form of a weight $w+2$ and $P$ a polynomial of degree $w$. The action of $GL^+(2,\mathbf{Z})$ is rather arbitrarily redistributed between $f$ and $P$.

To present this part of the structure more systematically, we have to consider four classes of linear representations (here understood as right actions) of $GL^+(2,\mathbf{Q})$: (i) the one–dimensional determinantal representation $g \mapsto \det g \cdot id$; (ii) the symmetric power of the basic two–dimensional representation; (iii) the "variable change" action upon $\mathcal{O}(H)_{cusp}$, that is, the inverse image $g^*$ with respect to the fractional linear action of $PGL^+(2,\mathbf{Q})$ $g:\ H \to H$, $g \mapsto g(z)$; (iv) the similar inverse image map on polydifferentials $\Omega_1(H)^{\otimes r}_{cusp}$, that is, holomorphic tensors $f(z)(dz)^r$, $r \in \mathbf{Z}$;

The latter action is traditionally translated into a "higher weight" action on functions $f(z)$ from $\mathcal{O}(H)_{cusp}$ via dividing by $(dz)^r$, and further tensoring by a power of the determinantal representation. As a result, the picture becomes somewhat messy, and moreover, in different expositions different normalizations and distributions of determinants between $f$ and $P$ are adopted. Anyway, we will fix our choices by the following conventions (the same as in [He1], [He2].)

For an integer $w \ge 0$, define the right action of weight $w+2$ upon holomorphic (or meromorphic) functions on $H$ by

$$f|[g]_{w+2}(z) := (\det g)^{w+1} f(gz) j(g,z)^{-(w+2)} \tag{2.17}$$

where we routinely denote $j(g,z) := cz + d$ for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Moreover, define the right action of $GL(2,\mathbf{R})$ on polynomials in two variables by

$$(Pg)(X,Y) := P((\det g)^{-1}(aX+bY), (\det g)^{-1}(cX+dY)). \tag{2.18}$$

Let now $\alpha$, $\beta$ be two points in $H \cup \mathbf{P}^1(\mathbf{Q})$. Then we have, for any homogeneous polynomial $P(X,Y)$ of degree $w$ and $g \in GL^+(2,\mathbf{Q})$, in view of (2.17) and (2.18),

$$\int_{g\alpha}^{g\beta} f(z)P(z,1)dz = \int_{\alpha}^{\beta} f(gz)P(gz,1)d(gz) =$$

$$\int_{\alpha}^{\beta} f|[g]_{w+2}(z) \cdot (\det g)^{-w-1} \cdot j(g,z)^{w+2} \cdot P\left(\frac{az+b}{cz+d}, 1\right) \cdot j(g,z)^{-2} \cdot \det g \cdot dz =$$

$$\int_{\alpha}^{\beta} f|[g]_{w+2}(z)P((\det g)^{-1}(az+b), (\det g)^{-1}(cz+d))dz =$$

$$\int_{\alpha}^{\beta} f|[g]_{w+2}(z)(Pg)(z,1)dz. \tag{2.19}$$

In particular, if for given $f$, $g$, and a constant $\varepsilon$ we have

$$f|[g]_{w+2}(z) = \varepsilon f(z), \tag{2.20}$$

then

$$\int_{g\alpha}^{g\beta} f(z)P(z,1)dz = \int_{\alpha}^{\beta} f(z)\varepsilon(Pg)(z,1)dz. \tag{2.21}$$

More generally, if for a finite family of $g_k \in GL^+(2,\mathbf{Q})$, $c_k \in \mathbf{C}$, we have

$$\sum_k c_k f|[g_k]_{w+2}(z) = \varepsilon f(z), \tag{2.22}$$

then

$$\sum_k c_k \int_{g_k\alpha}^{g_k\beta} f(z)P(z,1)dz = \int_{\alpha}^{\beta} f(z)\varepsilon \sum_k c_k (Pg_k)(z,1)dz. \tag{2.23}$$

These equations are especially useful in the standard context of modular and cusp forms and Hecke operators.

Let $\Gamma \subset SL(2, \mathbf{Z})$ be a subgroup of finite index, $w \geq 0$ an integer. Recall that a cusp form $f(z)$ of weight $w+2$ with respect to $\Gamma$ is a holomorphic function on the upper half–plane $H$, vanishing at cusps and such that for all $g \in \Gamma$, (2.20) holds with constant $\varepsilon = 1$. More generally, we can consider constants $\varepsilon = \chi(g)$, where $\chi$ is a character of $\Gamma$.

When $w$ is odd, nonvanishing cusp form can exist only if $-I \notin \Gamma$ where $I$ is the identity matrix. In particular, for $\Gamma = SL(2, \mathbf{Z})$ there can exist only cusp forms of even weight.

Denote by $S_{w+2}(\Gamma)$ (resp. $S_{w+2}(\Gamma, \chi)$) the complex space of cusp forms of weight $w+2$ for $\Gamma$ (resp. cusp forms with character $\chi$). Let $F_w$ be the space of polynomial forms of degree $w$ in two variables. Let $W$ be the space of linear functionals on $S_{w+2}(\Gamma) \otimes F_w$. Then the function on $(\alpha, \beta) \in \mathbf{P}^1(\mathbf{Q})^2$ with values in the space $W$

$$\mu(\alpha, \beta) : \, f \otimes P \mapsto \int_{\alpha}^{\beta} f(z) P(z, 1) dz \tag{2.24}$$

is a pseudo–measure.

We can call this pseudo–measure “the shadow” of the respective modular symbol.

Formulas (2.17) and (2.18) define the structure of a right $\Gamma$–module upon $S_{w+2}(\Gamma) \otimes F_w$ and hence the dual structure of a left $\Gamma$–module upon $W$. Formula (2.21) then shows that that the pseudo–measure (2.24) is modular with respect to $\Gamma$.

## 3. The Lévy functions and the Lévy–Mellin transform

**3.1. The Lévy functions.** A classical Lévy function (see [L]) $L(f)(\alpha)$ of a real argument $\alpha$ is given by the formula

$$L(f)(\alpha) := \sum_{n=0}^{\infty} f(q_n(\alpha), q_{n+1}(\alpha)) \tag{3.1}$$

where $q_n(\alpha)$, $n \geq 0$, is the sequence of denominators of normalized convergents to $\alpha$ (see (1.5)), and $f$ is a function defined on pairs $(q', q) \in \mathbf{Z}^2$, $1 \leq q' \leq q$, $\gcd(q', q) = 1$, taking values in a topological group and sufficiently quickly decreasing so that (3.1) absolutely converges. Then $L(f)(\alpha)$ is continuous on irrational numbers. Moreover, it has period 1.

It was remarked in [MaMar1] that for certain simple $f$ related to modular symbols, the integral $\int_0^1 L(f)(\alpha) d\alpha$ is a Dirichlet series directly related to the Mellin transform of an appropriate cusp form.

In this section, we will develop this remark and considerably generalize it in the context of modular pseudo–measures. We will call the involved integral representations *the Lévy–Mellin transform.* They are "shadows" of the classical Mellin transform.

As P. Lévy remarked in [L], for any $(q', q) \neq (1,1)$ as above all $\alpha \in [0, 1/2]$ such that $(q', q) = (q_n(\alpha), q_{n+1}(\alpha))$ fill a primitive semi–interval $I$ of length $(q(q+q'))^{-1}$. In $[0,1]$, such $\alpha$ fill in addition the semi–interval $1 - I$ (in [MaMar1], sec. 2.1, we have inadvertently overlooked this symmetry).

Therefore the class of functions $L(f)$ (restricted to irrational numbers) is contained in a more general class of (formal) infinite linear combinations of characteristic functions of primitive segments $I$:

$$L(f) := \sum_I f(I)\chi_I\,, \tag{3.2}$$

where $\chi_I(\alpha) = 1$ for $\alpha \in I$, 0 for $\alpha \notin I$.

A family of coefficients, i.e. a map $I \mapsto f(I)$ from the set of all primitive segments (positively oriented, or without a fixed orientation) in [0,1] to an abelian group $M$, will also be referred to as a Lévy function.

To treat this generalization systematically, we will display in the next subsection the relevant combinatorics of primitive segments.

**3.2. Various enumerations of primitive segments.** A matrix in $GL(2, \mathbf{Z})$ is called *reduced* if its entries are non–negative, and non–decreasing to the right in the rows and downwards in the columns. For a more thorough discussion, see [LewZa].

Clearly, each row and column of a matrix in $GL(2, \mathbf{Z})$, in particular, reduced ones, consists of co–prime entries. There is exactly one reduced matrix with lower row $(1, 1)$. This is one marginal case which does not quite fit in the pattern of the following series of bijections between several sets described below.

*The set $L$.* It consists of pairs $(c, d)$, $1 \le c < d \in \mathbf{Z}$, $gcd\,(c, d) = 1$.

*The set $R$.* It consists of pairs of reduced matrices $(g^-, g^+)$ with one and the same lower row and determinants, respectively, $-1$ and $+1$.

It is easy to see that for any pair $(c, d) \in L$, there exists exactly one pair $(g^-_{c,d}, g^+_{c,d}) \in R$ with lower row $(c, d)$ so that we have a natural bijection $L \to R$.

In the marginal case, only one reduced matrix $g^-_{1,1}$ exists.

*The set $S$.* It consists of pairs of primitive segments $(I^-, I^+)$ of length $< 1/2$ such that $I^- \subset [0, \dfrac{1}{2}]$, $I^+ = 1 - I^- \subset [\dfrac{1}{2}, 1]$. There is a well defined map $R \to S$

which produces from each pair $(g^-_{c,d}, g^+_{c,d}) \in R$ the pair

$$I^-_{c,d} := [g^-_{c,d}(0), g^-_{c,d}(1)], \quad I^+_{c,d} := [g^+_{c,d}(0), g^+_{c,d}(1)].$$

Again, it is a well defined bijection.

In the marginal case, we get one primitive segment $I^-_{1,1} = [0, \frac{1}{2}]$; it is natural to complete it by $I^+_{1,1} := [\frac{1}{2}, 1]$.

*The set $C$.* One element of this set is defined as a pair $(q', q) \in L$ such that there exists an $\alpha$, $0 < \alpha \leq \frac{1}{2}$, for which $(q', q)$ is a pair of denominators $(q_n(\alpha), q_{n+1}(\alpha))$ of two consecutive convergents to $\alpha$.

The sequence of consecutive convergents to a rational number $\alpha$ stabilizes at some $p_{n+1}/q_{n+1}$, so only a finite number of pairs $(q', q)$ is associated with $\alpha$. Integers, in particular 0 and 1, correspond to the marginal pair (1,1).

According to a lemma, used by P. Lévy in [L], all such $\alpha$ fill precisely the semi–interval $I^-_{q',q}$, with the end $g^-_{q',q}(1)$ excluded.

Moreover, those $\alpha$ which belong to the other half of $[0, 1]$ and as well admit $(q', q)$ as a pair of denominators of two consecutive convergents, fill the semi–interval $[g^+_{q',q}(0), g^+_{q',q}(1))$. This produces one more bijection $C = L \to S$, or rather an interpretation of a formerly constructed one.

The pairs of convergents involved are also encoded in this picture: they are $(g^-_{q',q}(0), g^-_{q',q}(\infty))$ and $(g^+_{q',q}(0), g^+_{q',q}(\infty))$ respectively.

Returning now to Lévy functions, we can rewrite (3.2) as

$$L(f)(\alpha) := f(I_{1,1}) + \sum_{I^-_{c,d} \ni \alpha} f(I^-_{c,d}) + \sum_{I^+_{c,d} \ni \alpha} f(I^+_{c,d}). \tag{3.3}$$

or else as a classical Lévy function

$$L(f)(\alpha) := f(1, 1) + \sum_{n \geq 0} f^-(q_n(\alpha), q_{n+1}(\alpha))\chi_{(0,1/2]}(\alpha) +$$

$$\sum_{n \geq 0} f^+(q_n(\alpha), q_{n+1}(\alpha))\chi_{[1/2,1)}(\alpha) \tag{3.4}$$

if the usual proviso about convergence is satisfied.

Notice that for any $(c, d) \in L$, the number $n$ such that $(c, d) = (q_n(\alpha), q_{n+1}(\alpha))$ is uniquely defined by $(c, d)$ and a choice of sign $\pm$ determining the position of $\alpha$ in the right/left half of $[0,1]$. The sequence of convergents preceding the $(n+1)$–th one is determined as well. This means that choosing $f^{\pm}$, we may explicitly refer to all these data, including the sequence of incomplete quotients up to the $(n+1)$–th one. Cf. examples in sec. 2.1 and 2.2.1 of [MaMar1].

**3.3. The Lévy–Mellin transform.** Below we will use formal Dirichlet series.

Let $\mathcal{A}$ be an abelian group. With any sequence $A=\{a_1,\dots,a_n,\dots\}$ of elements of $\mathcal{A}$ we associate a formal expression

$$L_A(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}\,.$$

If $\mathcal{B},\mathcal{C}$ are abelian groups, and we are given a composition $\mathcal{A}\times\mathcal{B}\to\mathcal{C} : (a,b)\mapsto a\cdot b$ which is a group homomorphism, we have the induced composition

$$(a_1,\dots,a_n,\dots)\cdot(b_1,\dots,b_n,\dots)=(c_1,\dots,c_n,\dots), \quad c_n := \sum_{d_1,d_2:\, d_1d_2=n} a_{d_1}\cdot b_{d_2}$$

which is interpreted as the multiplication of the respective formal Dirichlet series:

$$L_C(s)=L_A(s)\cdot L_B(s).$$

In particular, we can multiply any $L_A(s)$ by a Dirichlet series with integer coefficients.

Finally, the operator $\{a_1,\dots,a_n,\dots\}\mapsto\{1^w a_1, 2^w a_2,\dots,n^w a_n,\dots\}$ corresponds to the argument shift $L_A(s-w)$. Here generally $w\in\mathbf{Z}$, $w\ge 0$, but one can use also negative $w$ if $\mathcal{A}$ is a linear $\mathbf{Q}$–space.

Now let $W$ be a left $GL^+(2,\mathbf{Q})$–module: $w\mapsto g[w]$. We will apply this formalism to the Dirichlet series with coefficients in $\mathbf{Z}[GL^+(2,\mathbf{Q})]$, in $W$, and the composition induced by the above action.

Let $\mu\in M_W(SL(2,\mathbf{Z}))$ (for notation, see sec. 2.1.1.) Consider the following Lévy function $f_\mu$ non–vanishing only on the primitive segments in $[0,1/2]$:

$$f_\mu(I^-_{c,d})(s) := \frac{1}{|I^-_{c,d}|d^s}\begin{pmatrix}1 & -cd^{-1}\\ 0 & d^{-1}\end{pmatrix}\left[\mu(\infty,\frac{c}{d})\right]\,, \tag{3.5}$$

where the matrix preceding the square brackets acts upon $\mu(\infty,\frac{c}{d})$. This Lévy function takes values in the group of formal Dirichlet series with coefficients in $W$, with only one non–vanishing term of the series.

**3.3.1. Definition.** *The Mellin–Lévy transform of the pseudo–measure $\mu$ is the formal Dirichlet series with coefficients in $W$:*

$$LM_\mu(s) := \int_0^{1/2} L(f_\mu)(\alpha, s)d\alpha\,. \tag{3.6}$$

(The formal argument $s$ is included in the notation for future use.)

Moreover, introduce the following two formal Dirichlet series with coefficients in $\mathbf{Z}[GL^+(2,\mathbf{Z})]$:

$$Z_-(s) := \sum_{d_1=1}^{\infty} \begin{pmatrix} 1 & 0 \\ 0 & d_1^{-1} \end{pmatrix} \frac{1}{d_1^s}\,, \quad Z_+(s) := \sum_{d_2=1}^{\infty} \begin{pmatrix} d_2^{-1} & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{d_2^s}\,.$$

**3.4. Theorem.** *We have the following identity between the formal Dirichlet series:*

$$Z_+(s) \cdot Z_-(s) \cdot LM_\mu(s) = \sum_{n=1}^{\infty} \frac{(T_n\mu)(\infty, 0)}{n^s} \tag{3.7}$$

*where $T_n$ is the Hecke operator described in 2.5.3 and 2.5.2.*

**Proof.** Put $\overline{L} := L \cup \{(1,1)\}$, where the set $L$ was defined in sec. 3.2. Each matrix from (2.16) representing a coset in $SL(2,\mathbf{Z}) \setminus M_n(2,\mathbf{Z})$ can be uniquely written as

$$\begin{pmatrix} d_2 & cd_1 \\ 0 & dd_1 \end{pmatrix} \tag{3.8}$$

where $d, d_1, d_2$ are natural numbers with $dd_1d_2 = n$ and $(c,d) \in \overline{L}$. Acting from the left upon $(\infty, 0)$ this matrix produces $(\infty, cd^{-1})$. In the respective summand of $(T_n\mu)(\infty, 0)$ (cf. (2.15)), the inverse matrix to (3.8) acts on the left. We have

$$\begin{pmatrix} d_2 & cd_1 \\ 0 & dd_1 \end{pmatrix}^{-1} = \begin{pmatrix} d_2^{-1} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & d_1^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & -cd^{-1} \\ 0 & d^{-1} \end{pmatrix}\,. \tag{3.9}$$

Summing over all $d, d_1, d_2, c$ we get the identity (3.7) between the formal Dirichlet series.

This Theorem justifies the name of the Mellin–Lévy transform. In fact, applying (3.7) to a pseudo–measure associated with the space of $SL(2,\mathbf{Z})$–cusp forms of a fixed weight, and taking the value of the r.h.s. of (3.7) on an eigen–form for all Hecke operators, we get essentially the usual Mellin transform of this form. The l.h.s then furnishes its representation as an integral over a real segment (up to extra $Z_\pm$–factors) replacing the more common integral along the upper imaginary half–line.

Of course, one can establish versions of this theorem for the standard congruence subgroups $\Gamma_1(N)$, but (3.7) is universal in the same sense as the Fourier expansion presented in [Me] is universal.

**3.5.** ***p*–adic analogies continued.** Returning to the discussion in sec. 1.13, we now suggest the reader to compare the formula (3.5) with the definition of the $p$–adic measure given by the formula (39) of [Ma2].

We hope that this list makes convincing our suggestion that the theory of pseudo–measures can be considered as an $\infty$–adic phenomenon.

## 4. Non–commutative pseudo–measures and iterated Lévy–Mellin transform

**4.1. Definition.** *A pseudo–measure on* $\mathbf{P}^1(\mathbf{R})$ *with values in a non (necessarily) commutative group (written multiplicatively)* $U$ *is a function* $J:\mathbf{P}^1(\mathbf{Q})\times\mathbf{P}^1(\mathbf{Q})\to U$, $(\alpha,\beta)\mapsto J_\alpha^\beta\in U$ *satisfying the following conditions: for any* $\alpha,\beta,\gamma\in\mathbf{P}^1(\mathbf{Q})$,

$$J_\alpha^\alpha=1,\quad J_\beta^\alpha J_\alpha^\beta=1,\quad J_\gamma^\alpha J_\beta^\gamma J_\alpha^\beta=1\,. \tag{4.1}$$

The formalism of the sections 1 and 2 can be partially generalized to the non–commutative case.

**4.1.1. Identical and inverse pseudo–measures.** The set $M_U$ of $U$–valued pseudo–measures generally does not form a group, but only a set with involution and a marked point: $(J^{-1})_\alpha^\beta:=(J_\beta^\alpha)^{-1}$ and $J_\alpha^\beta\equiv 1_U$ are pseudo–measures.

**4.1.2. Universal pseudo–measure.** Let $\mathbf{U}$ be a free group freely generated by the set $\mathbf{Q}$. Let $\langle\alpha\rangle\in\mathbf{U}$ be the generator corresponding to $\alpha$. The map

$$(\alpha,\beta)\mapsto\langle\beta\rangle\langle\alpha\rangle^{-1},\quad(\infty,\beta)\mapsto\langle\beta\rangle\,,$$

is a pseudo–measure. It is universal in an evident sense (cf. sec. 1.4.)

**4.1.3. Primitive chains.** Let $I_j=(\alpha_j,\alpha_{j+1})$, $j=1,\dots,n$, be a primitive chain connecting $\alpha:=\alpha_1$ to $\beta:=\alpha_{n+1}$ as at the end of sec. 1.6. Then

$$J_\alpha^\beta=J_{\alpha_n}^{\alpha_{n+1}}J_{\alpha_{n-1}}^{\alpha_n}\dots J_{\alpha_1}^{\alpha_2}\,. \tag{4.2}$$

In particular, each pseudo–measure is determined by its values on primitive segments.

**4.2. Definition.** *An $U$–valued pre–measure $\widetilde{J}$ is a function on primitive segments satisfying the relations (4.1) written for primitive chains only.*

**4.3. Theorem.** Each pre–measure $\widetilde{J}$ can be uniquely extended to a pseudo–measure $J$.

The proof proceeds exactly as that of the Theorem 1.8, with only minor local modifications. We define $J$ by any of the formulas

$$J_\alpha^\beta = \widetilde{J}_{\alpha_n}^{\alpha_{n+1}} \widetilde{J}_{\alpha_{n-1}}^{\alpha_n} \dots \widetilde{J}_{\alpha_1}^{\alpha_2} . \tag{4.3}$$

using a primitive chain as in 4.1.3, and then prove that this prescription does not depend on the choice of this chain using the argument with elementary moves. One should check only that elementary moves are compatible with non–commutative relations (4.1) which is evident.

**4.4. Non–commutative reciprocity functions.** By analogy with 1.9, we can introduce the notion of a non–commutative $U$–valued reciprocity function $R_p^q$. The extension of the functional equation (1.13) should read

$$R_{p+q}^q R_p^{p+q} = R_p^q . \tag{4.4}$$

The analysis and results of 1.9–1.11 (not involving Dirichlet symbols) can be easily transported to this context.

**4.5. Right action of $GL^+(2, \mathbf{Q})$.** This group acts upon the set of pseudo–measures $M_U$ from the right as in the commutative case:

$$(Jg)_\alpha^\beta := J_{g\alpha}^{g\beta} .$$

**4.6. Modular pseudo–measures.** Assume now that a subgroup of finite index $\Gamma \subset SL(2, \mathbf{Z})$ acts upon $U$ from the left by group automorphisms, $u \mapsto gu$. As in the commutative case, we call an $U$–valued pseudo–measure $J$ *modular with respect to* $\Gamma$ if it satisfies the condition

$$(Jg)_\alpha^\beta = g[J_\alpha^\beta]$$

for all $g \in \Gamma$, $\alpha, \beta \in \mathbf{P}^1(\mathbf{Q})$. Denote by $M_U(\Gamma)$ the pointed set of such measures. We can repeat the argument and the construction of 2.1.2 showing that any element $J \in M_U(\Gamma)$ is uniquely determined by the values $J_{h_k(\infty)}^{h_k(0)} \in U$ where $\{h_k\}$ runs over a system of representatives of $\Gamma \setminus SL(2, \mathbf{Z})$.

An analog of 2.2 and of the Theorem 2.3 holds as well.

Namely, consider the map

$$M_U(SL(2,\mathbf{Z})) \to U : \ J \mapsto J^0_\infty .$$

From the argument above it follows that this is an injective map of pointed sets with involution. Its image is constrained by the similar conditions as in the abelian case.

Firstly, because of modularity, we have

$$(-id)[J^0_\infty] = J^0_\infty .$$

Therefore, the image of (3.1) is contained in $U_+$, $(-id)$–invariant subgroup of $U$. Hence we may and will consider this subgroup as a (non–commutative) module over $PSL(2,\mathbf{Z})$. Secondly,

$$\sigma[J^0_\infty] \cdot J^0_\infty = 1_U . \tag{4.5}$$

And finally,

$$\tau^2[J^0_\infty] \cdot \tau[J^0_\infty] \cdot J^0_\infty = 1_U . \tag{4.6}$$

To summarize, we get the following morphisms of pointed sets

$$M_U(SL(2,\mathbf{Z})) \to U_+ \to U_+ \times U_+ \tag{4.7}$$

where the first arrow is the embedding described above and the second one, say $\varphi$, sends $u$ to $(\sigma u \cdot u,\, \tau^2 u \cdot \tau u \cdot u)$.

**4.7. Theorem.** *The sequence of pointed sets (4.7) is exact. In other words, the map $J \mapsto J^0_\infty$ induces a bijection*

$$M_U(SL(2,\mathbf{Z})) \cong \varphi^{-1}(1_U, 1_U) \tag{4.8}$$

**Sketch of proof.** The proof follows exactly the same plan as that of Theorem 2.3. The only difference is that now the relations (1.1), (1.2) are replaced by their non–commutative versions (4.1), and the prescription (2.1) must be replaced by its non–commutative version based upon (4.2).

**4.8. Induced pseudo–measures.** As in the commutative case, by changing the group of values $U$, we can reduce the description of $M_U(\Gamma)$ for general $\Gamma$ to the case $\Gamma = SL(2,\mathbf{Z})$.

Namely, put

$$\widehat{U} := \mathrm{Map}_\Gamma(PSL(2,\mathbf{Z}), U) . \tag{4.9}$$

An element $\varphi \in \widehat{U}$ is thus a map $g \mapsto \varphi(g) \in U$ such that $\varphi(\gamma g) = \gamma\varphi(g)$ for all $\gamma \in \Gamma$ and $g \in SL(2,\mathbf{Z})$. Such functions form a group with pointwise multiplication, and $PSL(2,\mathbf{Z})$ acts on it via $(g\varphi)(\gamma) := \varphi(\gamma g)$.

Any $\widehat{U}$–valued $SL(2,\mathbf{Z})$–modular measure $\widehat{J}$ induces an $U$–valued $\Gamma$–modular measure $J$:

$$J_\alpha^\beta := \widehat{J}_\alpha^\beta(1_{SL(2,\mathbf{Z})})\,. \tag{4.10}$$

Conversely, any $U$–valued $\Gamma$–modular measure $J$ produces a $\widehat{U}$–valued $SL(2,\mathbf{Z})$–modular measure $\widehat{J}$:

$$\widehat{J}_\alpha^\beta(g) := J_{g(\alpha)}^{g(\beta)}\,. \tag{4.11}$$

**4.8.1. Proposition.** *The maps (4.10), (4.11) are well defined and mutually inverse. Thus, they produce a canonical isomorphism*

$$M_U(\Gamma) \cong M_{\widehat{U}}(SL(2,\mathbf{Z})). \tag{4.12}$$

The proof is straightforward.

**4.8.2. Pseudo–measures and cohomology.** As in the commutative case, given $J \in M_U(\Gamma)$ and $\alpha \in \mathbf{P}^1(\mathbf{Q})$, consider the function

$$c_\alpha^J = c_\alpha : \Gamma \to U, \quad c_\alpha(g) := J_{g\alpha}^\alpha.$$

From (4.1) and the modularity of $J$ it follows that this is a non–commutative 1–cocycle (we adopt the normalization as in [Ma3]):

$$c_\alpha(gh) = c_\alpha(g) \cdot gc_\alpha(h)\,.$$

Changing $\alpha$, we get a cohomologous cocycle:

$$c_\beta(g) = J_\alpha^\beta c_\alpha(g)\left(gJ_\alpha^\beta\right)^{-1}.$$

As in the commutative case, if we restrict $c_\alpha$ upon the subgroup $\Gamma_\alpha$ fixing $\alpha$, we get the trivial cocycle, so the respective cohomology class vanishes. This furnishes a canonical map of modular measures to the cuspidal cohomology

$$M_U(\Gamma) \to H^1(\Gamma, U)_{cusp}\,. \tag{4.13}$$

Again, for $\Gamma = PSL(2,\mathbf{Z})$ we have two independent descriptions of these sets, connected by the value $\mu(\infty, 0) = c_\infty(\sigma)$.

**4.8.3. Proposition.** *(i) For any $PSL(2,\mathbf{Z})$–modular pseudo–measure $J$, we have*

$$c_\infty^J(\sigma) = c_\infty^J(\tau) = J_0^\infty \in U_+ \,.$$

*(ii) This correspondence defines a bijection*

$$M_W(PSL(2,\mathbf{Z})) \cong Z^1(PSL(2,\mathbf{Z}), U_+)_{cusp}$$

*where the latter group by definition consists of cocycles with equal components $c(\sigma) = c(\tau)$.*

This easily follows along the lines of [Ma4], Proposition 1.2.1.

**4.9. Iterated integrals I.** In this and the next subsection we describe a systematic way to construct non–commutative pseudo–measures $J$ based upon iterated integration. We start with the classical case, when $J$ is obtained by integration along geodesics connecting cusps. This construction generalizes that of sec. 1.12 above, which furnishes the "linear approximation" to it.

Consider a pseudo–measure $\mu : \mathbf{P}^1(\mathbf{Q})^2 \to W$ given by the formulas (1.23) restricted to a finite–dimensional subspace $W^* \subset \mathcal{O}(H)_{cusp}$, that is, induced by the respective finite–dimensional quotient of $\mathbf{W}$ from the last paragraph of sec. 1.

Consider the ring of formal non–commutative series $\mathbf{C}\langle\langle W\rangle\rangle$ which is the completion of the tensor algebra of $W$ modulo powers of the augmentation ideal $(W)$. Put $U = 1 + (W)$. This is a multiplicative subgroup in this ring which will be the group of values of the following pseudo–measure $J$:

$$J_\alpha^\beta := 1 + \sum_{n=1}^\infty J_\alpha^\beta(n)\,. \tag{4.14}$$

Here $J_\alpha^\beta(n) \in W^{\otimes n}$ is defined as the linear functional upon $W^{*\otimes n}$ whose value at $f_1 \otimes \cdots \otimes f_n$, $f_i \in W^*$, is given by the iterated integral

$$J_\alpha^\beta(n)(f_1,\dots,f_n) := \int_\alpha^\beta f_1(z_1)\,dz_1 \int_\alpha^{z_1} f_2(z_2)\,dz_2 \cdots \int_\alpha^{z_{n-1}} f(z_n)\,dz_n\,. \tag{4.15}$$

The integration here is done over the simplex $\sigma_n(\alpha,\beta)$ consisting of the points $\beta > z_1 > z_2 > \dots z_n > \alpha$, the sign $<$ referring to the ordering along the geodesic oriented from $\alpha$ to $\beta$.

The basic properties of (4.14), including the pseudo–measure identities (4.1), are well known, cf. a review in sec. 1 of [Ma3]. In particular, all $J_\alpha^\beta$ belong to the subgroup of the so called *group–like* elements of $U$. This property compactly encodes all the *shuffle relations* between the iterated integrals (4.15).

Moreover, (4.13) is functorial with respect to the variable–change action of $GL^+(2,\mathbf{Q})$ so that if the linear term of (4.13), that is, the pseudo–measure (1.23), is $\Gamma$–modular, then $J$ is $\Gamma$–modular as well.

**4.10. Iterated integrals II.** We can now imitate this construction using iterated integrals of, say, piecewise continuous functions along segments in $\mathbf{P}^1(\mathbf{R})$ in place of geodesics. The formalism remains exactly the same, and the general results as well; of course, in this generality it has nothing to do with specifics of the situation we have been considering so far.

To re–introduce these specifics, we will iterate integrals of Lévy functions (3.1)–(3.4) and particularly integrands at the r.h.s. of (3.6). We will get in this way the "shadow" analogs of the more classical multiple Dirichlet series considered in [Ma3], and shuffle relations between them. It would be interesting to see whether one can in this way get new relations between the classical series. Here we only say a few words about the structure of the resulting series.

Consider an iterated integral of the form (4.15) in which $f_k(z)$ are now characteristic functions of finite segments $I_k$ in $\mathbf{P}^1(\mathbf{R})$. The following lemma is easy.

**4.10.1. Lemma.** *The integral (4.15) as a function of $\alpha, \beta$ and $2n$ ends of $I_1, \dots, I_n$ is a piecewise polynomial form of degree $n$. This form depends only on the relative order of all its arguments in $\mathbf{R}$.*

Consider now a $\Gamma$–modular $W$–valued pseudo–measure $\mu$ and a family of Lévy functions with coefficients in formal Dirichlet series as in (3.5). Then we can interpret an iterated integral of the form (4.15) with integrands $f_\mu(I^-_{c_k,d_k})(s_k)\cdot\chi_{I^-_{c_k,d_k}}(z)$, $k = 1, \dots, n$, as taking values directly in $W^{\otimes n}$. Accordingly, the iterated version of the Mellin–Lévy transform (3.6) will be represented by a formal Dirichlet–like series involving coefficients which are polynomial forms of the ends of primitive segments involved.

## 5. Pseudo–measures and limiting modular symbols

**5.1. Pseudo–measures and the disconnection space.** We recall the following definition of *analytic* pseudo–measures on totally disconnected spaces and their relation to currents on trees, cf. [vdP].

**5.1.1. Definition.** *Let $\Omega$ be a totally disconnected compact Hausdorff space and let $W$ be an abelian group. Let $C(\Omega, W) = C(\Omega,\mathbf{Z}) \otimes_{\mathbf{Z}} W$ denote the group of locally constant $W$–valued continuous functions on $\Omega$. An analytic $W$–valued pseudo–measure on $\Omega$ is a map $\mu : C(\Omega, W) \to W$ satisfying the properties:*

*(i)* $\mu(V \cup V') = \mu(V) + \mu(V')$ *for* $V, V' \subset \Omega$ *clopen subsets with* $V \cap V' = \emptyset$, *where we identify a set with its characteristic function.*

*(ii)* $\mu(\Omega) = 0$.

Equivalently, one can define analytic pseudo–measures as finitely additive functions on the Boolean algebra generated by a basis of clopen sets for the topology of $\Omega$ satisfying the conditions of Definition 5.1.1.

In particular, we will be interested in the case where the space $\Omega = \partial\mathcal{T}$ is the boundary of a tree. One defines currents on trees in the following way.

**5.1.2. Definition.** *Let* $W$ *be an abelian group and let* $\mathcal{T}$ *be a locally finite tree. We denote by* $\mathcal{C}(\mathcal{T}, W)$ *the group of* $W$*–valued currents on* $\mathcal{T}$*. These are* $W$*–valued maps* $\mathbf{c}$ *from the set of oriented edges of* $\mathcal{T}$ *to* $W$ *that satisfy the following properties:*

*(i) Orientation reversal:* $\mathbf{c}(\bar{e}) = -\mathbf{c}(e)$, *where* $\bar{e}$ *denotes the edge* $e$ *with the reverse orientation.*

*(ii) Momentum conservation:*

$$\sum_{s(e)=v} \mathbf{c}(e) = 0, \tag{5.1}$$

*where* $s(e)$ *(resp.* $t(e)$*) denote the source (resp. target) vertex of the oriented edge* $e$.

One can then identify currents on a tree with analytic pseudo–measures on its boundary, as in [vdP].

**5.1.3. Lemma.** *The group* $\mathcal{C}(\mathcal{T}, W)$ *of currents on* $\mathcal{T}$ *is canonically isomorphic to the group of* $W$*–valued finitely additive analytic pseudo–measures on* $\partial\mathcal{T}$*.*

**Proof.** The identification is obtained by setting

$$\mu(V(e)) = \mathbf{c}(e), \tag{5.2}$$

where $V(e) \subset \partial\mathcal{T}$ is the clopen subset of the boundary of $\mathcal{T}$ defined by all infinite admissible (i.e. without backtracking) paths in $\mathcal{T}$ starting with the oriented edge $e$.

The group of $W$–valued currents on $\mathcal{T}$ can also be characterized in the following way. We let $\mathcal{A}(\mathcal{T}, W)$ denote the $W$–valued functions on the oriented edges of $\mathcal{T}$ satisfying $\mu(\bar{e}) = -\mu(e)$ and let $C(\mathcal{T}^{(0)}, W)$ denote the $W$–valued functions on the set of vertices of $\mathcal{T}$. The following result is also proved in [vdP].

**5.1.4. Lemma.** *Let* $d : \mathcal{A}(\mathcal{T}, W) \to C(\mathcal{T}^{(0)}, W)$ *be given by*

$$d(f)(v) = \sum_{s(e)=v} f(e).$$

*Then the group of $W$–valued currents is given by $\mathcal{C}(\mathcal{T}, W) = Ker(d)$, so that one has an exact sequence*

$$0 \to \mathcal{C}(\mathcal{T}, W) \to \mathcal{A}(\mathcal{T}, W) \to C(\mathcal{T}^{(0)}, W) \to 0.$$

**5.2. The tree of $PSL(2, \mathbf{Z})$ and its boundary.** We will apply these results to the tree $\mathcal{T}$ of $PSL(2, \mathbf{Z})$ embedded in the hyperbolic plane $\mathbf{H}$. Its vertices are the elliptic points $\widetilde{I} \cup \widetilde{R}$, where $\widetilde{I}$ is the $PSL(2, \mathbf{Z})$–orbit of $i$ and $\widetilde{R}$ is the orbit of $\rho = e^{2\pi i/3}$. The set of edges is given by the geodesic arcs $\{\gamma(i), \gamma(\rho)\}$, for $\gamma \in SL(2, \mathbf{Z})$.

The relation between $\partial \mathcal{T}$ and $\mathbf{P}^1(\mathbf{R})$ can be summarized as follows.

**5.2.1. Lemma.** *The boundary $\partial \mathcal{T}$ is a compact Hausdorff space. There is a natural continuous $PSL(2, \mathbf{Z})$–equivariant surjection $\Upsilon : \partial \mathcal{T} \to \partial \mathbf{H} = \mathbf{P}^1(\mathbf{R})$, which is one–to–one on the irrational points $\mathbf{P}^1(\mathbf{R}) \cap (\mathbf{R} \smallsetminus \mathbf{Q})$ and two–to–one on rational points.*

**Proof.** Consider the Farey tessellation of the hyperbolic plane $\mathbf{H}$ by $PSL(2, \mathbf{Z})$ translates of the ideal triangle with vertices $\{0, 1, \infty\}$. The tree $\mathcal{T} \subset \mathbf{H}$ has a vertex of valence three in each triangle, a vertex of valence two bisecting each edge of the triangulation and three edges in each triangle joining the valence three vertex to each of the valence two vertices on the sides of the triangle.

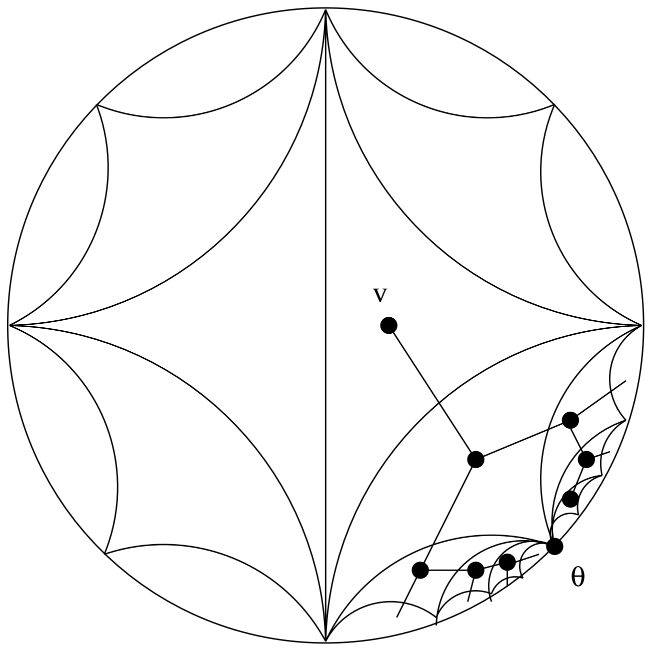
If we fix a base vertex in the tree, for instance the vertex $v = \rho = e^{2\pi i/3}$, then we can identify the boundary $\partial \mathcal{T}$ with the set of infinite admissible paths (i.e. paths without backtracking) in the tree $\mathcal{T}$ starting at $v$.

Any such path traverses an infinite number of triangles and can be encoded by the sequence of elements $\gamma_n \in PSL(2, \mathbf{Z})$ that determine the successive three–valent vertices crossed by the path, $v_n = \gamma_n v$.

This sequence of points $v_n \in \mathbf{H}$ accumulates at some point $\theta \in \mathbf{P}^1(\mathbf{R}) = \partial \mathbf{H}$. If the point is irrational, $\theta \in \mathbf{P}^1(\mathbf{R}) \smallsetminus \mathbf{P}^1(\mathbf{Q})$, then the point $\theta$ is not a vertex of any of the Farey triangles and there is a unique admissible sequence of vertices $v_n \in \tilde{R} \subset \mathbf{H}$ with the property that $\lim_{n\to\infty} v_n = \theta$. To see this, consider the family of segments $I_n \subset \mathbf{P}^1(\mathbf{R})$ given by all points that are ends of an admissible path in $\mathcal{T}$ starting at $v_n$ and not containing $v_{n-1}$.

The intersection $\cap_n I_n$ consists of a single point which is identified with the point in the limit set $\mathbf{P}^1(\mathbf{R})$ of $PSL(2, \mathbf{Z})$ given by the infinite sequence $\gamma_1 \gamma_2 \cdots \gamma_n \cdots$.

Consider now a rational point $\theta \in \mathbf{P}^1(\mathbf{Q})$. Then $\theta$ is a vertex of some (in fact infinitely many) of the Farey triangles. In this case, one can see that there are two distinct sequences of 3-valent vertices $v_n$ of $\mathcal{T}$ with the property that $\lim_{n\to\infty} v_n =$




$\theta$, due to the fact that, beginning with the first $v_k$ such that $\theta$ is a vertex of the triangle containing $v_k$, there are two adjacent triangles that also have $\theta$ as a vertex, see Figure 1.

This defines a map $\Upsilon : \partial\mathcal{T} \to \partial\mathbf{H}$, given by $\Upsilon(\{v_n\}) = \lim_{n\to\infty} v_n = \theta$. By construction it is continuous, $1:1$ on the irrationals and $2:1$ on the rationals.

**5.3. The disconnection space.** An equivalent description of the space $\partial\mathcal{T}$ of ends of the tree $\mathcal{T}$ of $PSL(2,\mathbf{Z})$ can be given in terms of the *disconnection* spaces considered in [Spi]. We discuss it here briefly, as it will be useful later in describing the noncommutative geometry of the boundary action of $PSL(2,\mathbf{Z})$.

Given a subset $U \subset \mathbf{P}^1(\mathbf{R})$ one considers the abelian $C^*$-algebra $\mathcal{A}_U$ generated by the algebra $C(\mathbf{P}^1(\mathbf{R}))$ and the characteristic functions of the positively oriented intervals (in the sense of §1.1 above) with endpoints in $U$. If the set $U$ is dense in $\mathbf{P}^1(\mathbf{R})$ then this is the same as the closure in the supremum norm of the $*$-algebra generated by these characteristic functions.

The Gelfand–Naimark correspondence $X \leftrightarrow C_0(X)$ furnishes an equivalence of categories of locally compact Hausdorff topological spaces $X$ and commutative $C^*$–algebras respectively. Thus, we have $\mathcal{A}_U = C(D_U)$, where the topological space $D_U$ is called *the disconnection* of $\mathbf{P}^1(\mathbf{R})$ along $U$. It is a compact Hausdorff space and it is totally disconnected if and only if $U$ is dense in $\mathbf{P}^1(\mathbf{R})$.

In particular, one can consider the set $U = \mathbf{P}^1(\mathbf{Q})$ and the resulting disconnection $D_{\mathbf{Q}}$ of $\mathbf{P}^1(\mathbf{R})$ along $\mathbf{P}^1(\mathbf{Q})$.

**5.3.1. Lemma.** *The map $\Upsilon : \partial\mathcal{T} \to \mathbf{P}^1(\mathbf{R})$ of Lemma 5.2.1 factors through a homeomorphism $\widetilde{\Upsilon} : \partial\mathcal{T} \to D_{\mathbf{Q}}$, followed by the surjective map $D_{\mathbf{Q}} \to \mathbf{P}^1(\mathbf{R})$ determined by the inclusion of the algebra $C(\mathbf{P}^1(\mathbf{R}))$ inside $C(D_{\mathbf{Q}})$.*

**Proof.** The compact Hausdorff space $\partial\mathcal{T}$ is dual to the abelian $C^*$-algebra $C(\partial\mathcal{T})$. After the choice of a base vertex $v \in \mathcal{T}$, the topology of $\partial\mathcal{T}$ is generated by the clopen sets $V(v')$, of ends of admissible paths starting at a vertex $v'$ and not passing through $v$.

In fact, it suffices to consider only 3–valent vertices, because for a vertex $v'$ of valence two we have $V(v') = V(v'')$ with $v''$ being the next 3–valent vertex in the direction away from $v$. The $C^*$–algebra $C(\partial\mathcal{T})$ is thus generated by the characteristic functions of the $V(v')$, since $\partial\mathcal{T}$ is a totally disconnected space. Since $v' = \gamma v$ for some $\gamma \in PSL(2,\mathbf{Z})$, the set $V(v') = \Upsilon^{-1}(I(v')) \subset \partial\mathcal{T}$, where $I(v') \subset \mathbf{P}^1(\mathbf{R})$ is a segment of the form $I(v') = \gamma I$, where $I$ is one of the segments $[\infty, 0]$ or $[0,1]$ or $[1,\infty]$. These are the segments in $\mathbf{P}^1(\mathbf{R})$ of the form $[p/q, r/s]$ for $p,q,r,s \in \mathbf{Z}$ with $ps - qr = \pm 1$, that is, the primitive segments on the boundary, as defined in §1.6, corresponding to sides of triangles of the Farey tessellation.

On the other hand the disconnection $D_{\mathbf{Q}}$ of $\mathbf{P}^1(\mathbf{R})$ along $\mathbf{P}^1(\mathbf{Q})$ is dual to the abelian $C^*$-algebra generated by all the characteristic functions of the oriented intervals with endpoints in $\alpha, \beta \in \mathbf{P}^1(\mathbf{Q})$. Since the sets $V(e)$ that generate the topology of $\partial\mathcal{T}$ are of this form, this shows that there is an injection $C(\partial\mathcal{T}) \to C(D_{\mathbf{Q}})$. The primitive intervals of §1.6 in fact give a basis for the topology of $D_{\mathbf{Q}}$, since the characteristic functions that generate $C(D_{\mathbf{Q}})$ can be written as combinations of characteristic functions of such intervals, using primitive chains as in §1.6. This shows that the map is in fact an isomorphism.

**5.4. Analytic pseudo–measures on $D_{\mathbf{Q}}$.** We show here that the pseudo–measures on $\mathbf{P}^1(\mathbf{R})$ (Definition 2.1.1) can be regarded as analytic pseudo–measures on the disconnection space $D_{\mathbf{Q}}$.

**5.4.1. Lemma.** *There is a natural bijection between $W$–valued pseudo–measures on $\mathbf{P}^1(\mathbf{R})$ and $W$–valued analytic pseudo–measures on $D_{\mathbf{Q}}$ (or equivalently, $W$–valued currents on $\mathcal{T}$.)*

**Proof.** Recall that a basis for the topology of $D_{\mathbf{Q}}$ is given by the preimages under the map $\Upsilon$ of the primitive (Farey) segments in $\mathbf{P}^1(\mathbf{R})$. These are segments of the form $(g(\infty), g(0))$ for some $g \in G = PSL(2,\mathbf{Z})$. Let $e$ be an oriented edge in the tree $\mathcal{T}$ of $PSL(2,\mathbf{Z})$. A basis of the topology of $\partial\mathcal{T}$ is given by sets of the form $V(e)$. These are in fact Farey intervals and all such intervals arise in this way. Thus, an analytic pseudo–measure on $D_{\mathbf{Q}}$ is a finitely additive function on the Boolean algebra generated by the Farey intervals with the properties of Definition 5.1.1.

A pseudo–measure on $\mathbf{P}^1(\mathbf{R})$, on the other hand, is a map $\mu : \mathbf{P}^1(\mathbf{Q}) \times \mathbf{P}^1(\mathbf{Q}) \to W$ with the properties (1.1) and (1.2). Such a function in fact extends, as we have seen, to a finitely additive function on the Boolean algebra consisting of finite unions of (positively oriented) intervals $(\alpha, \beta)$ with $\alpha, \beta \in \mathbf{P}^1(\mathbf{Q})$.

We then just set $\mu_{an}(V(e)) := \mu(\alpha, \beta)$, where $\mathbf{P}^1(\mathbf{R}) \supset [\alpha, \beta] = \Upsilon(V(e))$. This is then a finitely additive function on the Boolean algebra of the $V(e)$. The condition $\mu(\beta, \alpha) = -\mu(\alpha, \beta)$ implies that $\mu_{an}(\partial\mathcal{T}) = 0$ and similarly this combined with the property $\mu(\alpha, \beta) + \mu(\beta, \gamma) + \mu(\gamma, \alpha) = 0$ implies that $\mu_{an}(V \cup V') = \mu_{an}(V) + \mu_{an}(V')$ for $V \cap V' = \emptyset$.

Conversely, start with an analytic pseudo–measure $\mu_{an}$ on $D_{\mathbf{Q}}$ with values in $W$. For $\Upsilon V(e) = [\alpha, \beta]$ we write $\mu(\alpha, \beta) := \mu_{an}(V(e))$. Then the properties $\mu_{an}(V \cup V') = \mu_{an}(V) + \mu_{an}(V')$ for $V \cap V' = \emptyset$ and $\mu_{an}(\partial\mathcal{T}) = 0$ imply the corresponding properties $\mu(\beta, \alpha) = -\mu(\alpha, \beta)$ and $\mu(\alpha, \beta) + \mu(\beta, \gamma) + \mu(\gamma, \alpha) = 0$ for the pseudo–measure.

Under this correspondence, $\Gamma$–modular $W$–valued pseudo–measures correspond to $\Gamma$–modular analytic $W$–valued pseudo–measures. Thus, in the following we often use the term pseudo–measure equivalently for the analytic ones on $D_{\mathbf{Q}}$ or for those defined in §2 on $\mathbf{P}^1(\mathbf{R})$.

**5.5. K–theoretic interpretation.** In [MaMar1] we gave an interpretation of the modular complex of [Ma1] in terms of $K$-theory of $C^*$-algebras, by considering the crossed product $C^*$-algebra of the action of $G = PSL(2, \mathbf{Z})$ on its limit set $\mathbf{P}^1(\mathbf{R})$. Here we consider instead the action of $PSL(2, \mathbf{Z})$ on the ends $\partial\mathcal{T} = D_{\mathbf{Q}}$ of the tree and we obtain a similar $K$-theoretic interpretation of pseudo–measures.

More generally, let $\Gamma \subset PSL(2, \mathbf{Z})$ be a finite index subgroup and we consider the action of $G = PSL(2, \mathbf{Z})$ on $\partial\mathcal{T} \times \mathbf{P}$, where $\mathbf{P} = \Gamma \backslash PSL(2, \mathbf{Z})$. We let $\mathcal{A} = C(\partial\mathcal{T} \times \mathbf{P})$ with the action of $G = PSL(2, \mathbf{Z}) = \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$ by automorphisms.

**5.5.1. Lemma.** *The $K$–groups of the crossed product $C^*$–algebra $\mathcal{A} \rtimes G$ have the following structure:*

$$K_0(\mathcal{A} \rtimes G) = Coker(\alpha), \quad K_1(\mathcal{A} \rtimes G) = Ker(\alpha),$$

*where*

$$\alpha : C(\partial\mathcal{T} \times \mathbf{P}, \mathbf{Z}) \to C(\partial\mathcal{T} \times \mathbf{P}, \mathbf{Z})^{G_2} \oplus C(\partial\mathcal{T} \times \mathbf{P}, \mathbf{Z})^{G_3}$$

*is given by $\alpha : f \mapsto (f + f \circ \sigma, f + f \circ \tau + f \circ \tau^2)$ for $\sigma$ and $\tau$, respectively, the generators of $G_2$ and $G_3$ defined by (2.4).*

**Proof.** First recall that, for a totally disconnected space $\Omega$, one can identify the locally constant integer valued functions $C(\Omega, \mathbf{Z})$ with $K_0(C(\Omega))$, whereas $K_1(C(\Omega)) = 0$.

The six–terms exact sequence of [Pim] for groups acting on trees gives

$$0 \to K_1(\mathcal{A} \rtimes G) \to K_0(\mathcal{A}) \to K_0(\mathcal{A} \rtimes G_2) \oplus K_0(\mathcal{A} \rtimes G_3) \to K_0(\mathcal{A} \rtimes G) \to 0,$$

for $G = PSL(2, \mathbf{Z})$ and $G_i$ equal to $\mathbf{Z}/2\mathbf{Z}$ or $\mathbf{Z}/3\mathbf{Z}$. Let $\alpha : K_0(\mathcal{A} \rtimes G_2) \oplus K_0(\mathcal{A} \rtimes G_3) \to K_0(\mathcal{A} \rtimes G)$ be the map in the sequence above.

Here we use the fact that $K_1(\mathcal{A}) = 0$ and $K_1(\mathcal{A} \rtimes G_i) = K^1_{G_i}(\mathcal{A}) = 0$ so that the remaining terms in the six–terms exact sequence do not contribute.

Moreover, we have $K_0(\mathcal{A}) = C(\partial \mathcal{T} \times \mathbf{P}, \mathbf{Z})$. Similarly, we have

$$K_0(\mathcal{A} \rtimes G_i) = K^0_{G_i}(\partial \mathcal{T} \times \mathbf{P}) = C(\partial \mathcal{T} \times \mathbf{P}, \mathbf{Z})^{G_i}.$$

Following [Pim], we see that the map $\alpha$ can be described as the map

$$\alpha : f \mapsto (f + f \circ \sigma, f + f \circ \tau + f \circ \tau^2).$$

This gives a description of $Ker(\alpha)$ as

$$Ker(\alpha) = \{f \in C(\partial \mathcal{T} \times \mathbf{P}, \mathbf{Z}) \,|\, f + f \circ \sigma = f + f \circ \tau + f \circ \tau^2 = 0\}.$$

Similarly, the cokernel of $\alpha$ is the group of coinvariants, that is, the quotient of $C(\partial \mathcal{T} \times \mathbf{P}, \mathbf{Z})$ by the submodule generated by the elements of the form $f + f \circ \sigma$ and $f + f \circ \tau + f \circ \tau^2$. By the six–terms exact sequence we know that $K_1(\mathcal{A} \rtimes G) = Ker(\alpha)$ and $K_0(\mathcal{A} \rtimes G) = Coker(\alpha)$.

For simplicity let us reduce to the case with $\mathbf{P} = \{1\}$, that is, $\Gamma = G = PSL(2, \mathbf{Z})$.

**5.5.2. Integral.** Given a $W$–valued pseudo–measure $\mu$ on $D_{\mathbf{Q}}$, in the sense of Definition 5.1.1, we can define an integral

$$f \mapsto \int f \, d\mu \in W,$$

for $f \in C(D_{\mathbf{Q}}, \mathbf{Z})$ in the following way.

An element $f \in C(D_{\mathbf{Q}}, \mathbf{Z})$ is of the form $f = \sum_{i=1}^n a_i \chi_{I_i}$ with $a_i \in \mathbf{Z}$ and the intervals $I_i \subset D_{\mathbf{Q}}$ with $\Upsilon I_i = g_i(\infty, 0) \subset \mathbf{P}^1(\mathbf{R})$, for some $g_i \in G$. Thus, a natural prescription is

$$\int f \, d\mu = \sum_i a_i \mu(I_i) \in W.$$

To simplify notation, in the following we often do not distinguish between an interval in $\mathbf{P}^1(\mathbf{R})$ and its lift to the disconnection space $D_{\mathbf{Q}}$. So we write equivalently $\mu(I_i)$ or $\mu(g_i(\infty), g_i(0))$.

**5.5.3. Lemma.** *Let $g \in G$. The following change of variable formula holds:*

$$\int f \circ g \, d\mu = \int f \, d(\mu \circ g^{-1}).$$

**Proof.** We have $f \circ g = \sum_i a_i \chi_{I_i} \circ g = \sum_i a_i \chi_{g^{-1} I_i}$, so that

$$\int f \circ g \, d\mu = \sum_i a_i \mu(g^{-1} g_i(\infty), g^{-1} g_i(0))$$

$$= \sum_i a_i \mu \circ g^{-1}(g_i(\infty), g_i(0)) = \int f \, d(\mu \circ g^{-1}).$$

The following generalization is also true (and easy):

$$\int (f \circ g) \, h \, d\mu = \int f \, (h \circ g^{-1}) d(\mu \circ g^{-1}),$$

for $f, h \in C(D_{\mathbf{Q}}, \mathbf{Z})$.

**5.5.4. Proposition.** *Let $\mu$ be a $G$-modular $W$–valued pseudo–measure. For any $h \in K_1(\mathcal{A} \rtimes G)$, there exists a unique $G$-modular $W$-valued pseudomeasure $\mu_h$ with*

$$\mu_h(\infty, 0) := \int h \, d\mu$$

**Proof.** We will consider the case of $G = PSL(2, \mathbf{Z})$. The general case can be reduced to this one by proceeding as in 2.4 for the modular pseudomeasures.

In view of the Theorem 2.3, it suffices to check that the element

$$\int h \, d\mu \in W,$$

is annihilated by $1 + \sigma$ and $1 + \tau + \tau^2$.

An element $h \in K_1(\mathcal{A} \rtimes G)$ is a function $f \in C(\partial \mathcal{T}, \mathbf{Z})$ satisfying $h + h \circ \sigma = 0$ and $h + h \circ \tau + h \circ \tau^2 = 0$. Therefore

$$(1 + \sigma) \int h \, d\mu = \int h \, d\mu + \int h \, d\mu \circ \sigma = \int (h + h \circ \sigma) d\mu = 0.$$

Similarly,

$$(1+\tau+\tau^2)\int h\,d\mu = \int (h + h\circ\tau^2 + h\circ\tau)d\mu = 0.$$

Thus, one obtains a $G$-modular pseudo-measure $\mu_h$ for each $h \in K_1(\mathcal{A} \rtimes G)$.

**5.5.5. Proposition.** *A $G$-modular $W$–valued pseudo–measure $\mu$ defines a group homomorphism $\mu : K_0(\mathcal{A} \rtimes G) \to W$ induced by integration with respect to $\mu$.*

**Proof.** Using the identification with analytic pseudomeasures, we can consider the functional from $C(\partial\mathcal{T}, \mathbf{Z})$ to $W$ given by integration

$$\mu(f) = \int f\,d\mu.$$

This descends to the quotient of $C(\partial\mathcal{T}, \mathbf{Z})$ by the relations $f + f\circ\sigma$ and $f + f\circ\tau + f\circ\tau^2$. In fact, it suffices to consider the case where $f$ is the characteristic function of a segment $(g(\infty), g(0))$. We have

$$\int (f + f\circ\sigma)d\mu = \int f\,d\mu + \int f\,d\mu\circ\sigma = (1+\sigma)\int f\,d\mu$$

and

$$\int \chi_{(g(\infty),g(0))}\,d\mu = \int \chi_{(\infty,0)}\circ g^{-1}\,d\mu = \int \chi_{(\infty,0)}\,d\mu\circ g$$

so that

$$\sigma\int \chi_{(\infty,0)}\,d\mu\circ g = \int \chi_{(\infty,0)}d\mu\circ g\circ\sigma = \int \chi_{(g\sigma(\infty),g\sigma(0))}\,d\mu,$$

hence

$$(1+\sigma)\int \chi_{(g(\infty),g(0))}\,d\mu = g(1+\sigma)\mu(\infty,0) = 0.$$

The argument for $f + f\circ\tau + f\circ\tau^2$ is similar.

**5.6. Boundary action and noncommutative spaces.** In [MaMar1], we described the noncommutative boundary of the modular tower in terms of the quotient $\Gamma\backslash\mathbf{P}^1(\mathbf{R})$, which we interpreted as a noncommutative space, described by the crossed product $C^*$-algebra $C(\mathbf{P}^1(\mathbf{R}))\rtimes\Gamma$ or $C(\mathbf{P}^1(\mathbf{R})\times\mathbf{P})\rtimes G$, for $\Gamma\subset G$ a finite index subgroup and $\mathbf{P}$ the coset space.

Here we have seen that, instead of considering $\mathbf{P}^1(\mathbf{R})$ as the boundary, one can also work with the disconnection space $D_{\mathbf{Q}}$. We have then considered the corresponding crossed product $C(D_{\mathbf{Q}} \times \mathbf{P}) \rtimes G$. This is similar to the treatment of Fuchsian groups described in [Spi].

There is, however, another way of describing the boundary action of $G = PSL(2,\mathbf{Z})$ on $\mathbf{P}^1(\mathbf{R})$. As we have also discussed at length in [MaMar1], it uses a dynamical system associated to the Gauss shift of the continued fraction expansion, generalized in the case of $\Gamma \subset G$ to include the action on the coset space $\mathbf{P}$. In [MaMar1] we worked with $PGL(2,\mathbf{Z})$ instead of $PSL(2,\mathbf{Z})$. For the $PSL(2,\mathbf{Z})$ formulation, see [ChMay], [Mayer] and [KeS].

This approach to describing the boundary geometry was at the basis of our extension of modular symbols to limiting modular symbols. We discuss briefly here how this formulation also leads to a noncommutative space, in the form of an Exel–Laca algebra.

**5.6.1. The generalized Gauss shift.** The Gauss shift for $PSL(2,\mathbf{Z})$ is the map $\hat{T} : [-1,1] \to [-1,1]$ of the form

$$\widehat{T} : x \mapsto -sign(x)T(|x|),$$

where

$$T(x) = \frac{1}{x} - \left[\frac{1}{x}\right].$$

Notice that this differs from the Gauss shift $T : [0,1] \to [0,1]$, $T(x) = 1/x - [1/x]$ of $PGL(2,\mathbf{Z})$ by the presence of the extra sign, as in [ChMay], [KeS].

When one considers a finite index subgroup $\Gamma \subset G = PSL(2,\mathbf{Z})$ one extends the shift $\hat{T}$ to the generalized Gauss shift

$$\hat{T}_{\mathbf{P}} : \mathcal{I} \times \mathbf{P} \to \mathcal{I} \times \mathbf{P}, \quad (x,s) \mapsto (\hat{T}(x), [gST^k]),$$

where here $\mathcal{I} = [-1,1] \cap (\mathbf{R} \smallsetminus \mathbf{Q})$ and where $k = sign(x)n_1$. Here $\mathbf{P} = \Gamma\backslash G$ is the coset space and $g \in G$ denotes the representative $\Gamma g = s \in \mathbf{P}$. The $n$–th iterate of the map $\hat{T}_{\mathbf{P}}$ acts on $\mathbf{P}$ as the $SL(2,\mathbf{Z})$ matrix

$$\begin{pmatrix} -sign(x_1)p_{k-1}(x) & (-1)^k p_k(x) \\ q_{k-1}(x) & (-1)^{k+1} q_k(x) \end{pmatrix}.$$

**5.6.2. Shift happens.** Instead of considering the action of the generalized Gauss shift on the space $\mathcal{I} \times \mathbf{P}$, one can proceed as in [KeS] and introduce a shift space over which $\hat{T}_{\mathbf{P}}$ acts as a shift operator. This is obtained by considering the countable alphabet $\mathbf{Z}^{\times} \times \mathbf{P}$ and the set of *admissible sequences*

$$\Sigma_\Gamma = \{((x_1,s_1),(x_2,s_2),\dots) \mid A_{(x_i,s_i),(x_{i+1},s_{i+1})} = 1\},$$

where the matrix $A$ giving the admissibility condition is defined as follows. One has $A_{(x,s),(x',s')} = 1$ if $xx' < 0$ and

$$s' = \tau_x(s) := [gST^{x_1}] \in \mathbf{P}, \quad \text{where } s = \Gamma g$$

and $A_{(x,s),(x',s')} = 0$ otherwise. The action of $\widehat{T}_{\mathbf{P}}$ described above becomes the action of the one–sided shift $\sigma : \Sigma_\Gamma \to \Sigma_\Gamma$

$$\sigma : ((x_1, s_1), (x_2, s_2), \dots) \mapsto ((x_2, s_2), (x_3, s_3), \dots).$$

The space $\Sigma_\Gamma$ can be topologized in the way that is customarily used to treat this type of dynamical systems given by shift spaces. Namely, one considers on $\Sigma_\Gamma$ the topology generated by the cylinders (all words in $\Sigma_\Gamma$ starting with an assigned finite admissible word in the alphabet). This makes $\Sigma_\Gamma$ into a compact Hausdorff space. One can see, in fact, that the topology is metrizable and induced for instance by the metric

$$d((x_k, s_k)_k, (x'_k, s'_k)_k) = \sum_{n=1}^{\infty} 2^{-n} \left(1 - \delta_{(x_n,s_n),(x'_n,s'_n)}\right).$$

As is shown in [KeS], the use of this topology as opposed to the one induced by $\mathbf{P}^1(\mathbf{R}) \times \mathbf{P}$ simplifies the analysis of the associated Perron–Frobenius operator. The latter now falls into the general framework developed in [MaUr] and one obtains the existence of a shift invariant ergodic measure on $\Sigma_\Gamma$ from this general formalism. One uses essentially the *finite irreducibility* of the shift space $(\Sigma_\Gamma, \sigma)$, which follows in [KeS] from the ergodicity of the geodesic flow. The $\sigma$–invariant measure on $\Sigma_\Gamma$ induces via the bijection between these spaces a $\hat{T}_{\mathbf{P}}$–invariant measure on the space $\mathcal{I} \times \mathbf{P}$.

**5.6.3. Exel–Laca algebras.** There is a way to associate to a shift space on an alphabet a noncommutative space, in the form of a Cuntz–Krieger algebra in the case of a finite alphabet [CuKrie], or more generally an Exel–Laca algebra for a countable alphabet [ExLa].

Start with a (finite or countable) alphabet $\mathbf{A}$ and a matrix $A : \mathbf{A} \times \mathbf{A} \to \{0, 1\}$ that assigns the admissibility condition for words in the alphabet $\mathbf{A}$. In the case of a finite alphabet $\mathbf{A}$, the corresponding Cuntz–Krieger algebra is the $C^*$-algebra generated by partial isometries $S_a$ for $a \in \mathbf{A}$ with the relations

$$S_a S_a^* = P_a, \quad \text{with } \sum_a P_a = 1 \tag{5.3}$$

$$S_a^* S_a = \sum_{b \in \mathbf{A}} A_{ab} S_b S_b^*. \tag{5.4}$$

In the case of a countably infinite alphabet $\mathbf{A}$, one has to be more careful, as the summations that appear in the relations (5.3) and (5.4) no longer converge in norm. A version of CK algebras for infinite matrices was developed by Exel and Laca in [ExLa]. One modifies the relations (5.3) and (5.4) in the following way.

**5.6.4. Definition ([ExLa]).** *For a countably infinite alphabet $\mathbf{A}$ and an admissibility matrix $A : \mathbf{A} \times \mathbf{A} \to \{0, 1\}$, the CK algebra $O_A$ is the universal $C^*$-algebra generated by partial isometries $S_a$, for $a \in \mathbf{A}$, with the following conditions:*

*(i) $S_a^* S_a$ and $S_b^* S_b$ commute for all $a, b \in \mathbf{A}$.*

*(ii) $S_a^* S_b = 0$ for $a \neq b$.*

*(iii) $(S_a^* S_a) S_b = A_{ab} S_b$ for all $a, b \in \mathbf{A}$.*

*(iv) For any pair of finite subsets $X, Y \subset \mathbf{A}$, such that the product*

$$A(X, Y, b) := \prod_{x \in X} A_{xb} \prod_{y \in Y} (1 - A_{yb}) \tag{5.5}$$

*vanishes for all but finitely many $b \in \mathbf{A}$, one has the identity*

$$\prod_{x \in X} S_x^* S_x \prod_{y \in Y} (1 - S_y^* S_y) = \sum_{b \in \mathbf{A}} A(X, Y, b) S_b S_b^*. \tag{5.6}$$

The conditions listed above are obtained by formal manipulations from the relations (5.3) and (5.4) and are equivalent to them in the finite case.

**5.6.5. The algebra of the generalized Gauss shift.** We now consider the shift space $(\Sigma_\Gamma, \sigma)$ of [KeStr]. In this case we have the alphabet $\mathbf{A} = \mathbf{Z}^\times \times \mathbf{P}$ and the admissibility matrix given by the condition

$$A_{ab} = 1 \quad \text{iff } nn' < 0 \quad \text{and} \quad s = s' \, ST^{n'},$$

for $a = (n, s)$ and $b = (n', s')$ in $\mathbf{A}$. The corresponding Exel–Laca algebra $O_A$ is generated by isometries $S_{(n,s)}$ satisfying the conditions of Definition 5.5.4 above.

Let $\mathcal{J} = \Upsilon^{-1}[-1, 1] \subset D_{\mathbf{Q}}$ be the preimage of the interval $[-1, 1]$ under the continuous surjection $\Upsilon : D_{\mathbf{Q}} \to \mathbf{P}^1(\mathbf{R})$ and let $\mathcal{J}_{+1} = \Upsilon^{-1}[0, 1]$ and $\mathcal{J}_{-1} = \Upsilon^{-1}[-1, 0]$. Also let

$$\mathcal{J}_k = \{x \in \mathcal{J} \mid \Upsilon(x) = sign(k)[a_1, a_2, \ldots],\ a_1 = |k|,\ a_i \geq 1\}.$$

Let $\Gamma$ be a finite index subgroup of $G = PSL(2, \mathbf{Z})$, $\mathbf{P} = \Gamma\backslash G$. Consider the sets

$$\mathcal{J}_{k,s} := \mathcal{J}_k \times \{s\},$$

for $s \in \mathbf{P}$ and for $k \in \mathbf{Z}^\times$. Let $\chi_{k,s}$ denote the characteristic function of the set $\mathcal{J}_{k,s}$.

**5.6.6. Proposition.** *The subalgebra of the crossed product $C(D_{\mathbf{Q}} \times \mathbf{P}) \rtimes G$ generated by elements of the form*

$$S_{k,s} := \chi_{k,s} U_k,$$

*where $U_k = U_{\gamma_k}$, for $\gamma_k = T^k S \in \Gamma$, is isomorphic to the Exel–Laca algebra $O_A$ of the shift $(\Sigma_\Gamma, \sigma)$.*

**Proof.** We need to check that the $S_{k,s}$ satisfy the Exel–Laca axioms of Definition 5.5.4. We have $S^*_{k,s} = U^*_k \chi_{k,s} = U^*_k(\chi_{k,s}) U^*_k$ so that $S_{k,s} S^*_{k,s} = \chi_{k,s} U_k U^*_k \chi_{k,s} = \chi_{k,s} =: P_{k,s}$ and $S^*_{k,s} S_{k,s} = U^*_k \chi_{k,s} \chi_{k,s} U_k = U^*_k(\chi_{k,s})$. One sees from this that $S^*_{k,s} S_{k,s}$ and $S^*_{k',s'} S_{k',s'}$ commute, as they both belong to the subalgebra $C(D_{\mathbf{P}^1(\mathbf{Q})} \times \mathbf{P})$, so that the first axiom is satisfied. Similarly, $S^*_{k,s} S_{k',s'} = U^*_k \chi_{k,s} \chi_{k',s'} U_{k'} = 0$ for $(k,s) \neq (k',s')$, which is the second axiom. To check the third axiom, we have

$$(S^*_{k,s} S_{k,s}) S_{k',s'} = U^*_k(\chi_{k,s}) \chi_{k',s'} U_{k'}.$$

We now describe more explicitly the element $U^*_k(\chi_{k,s})$. Since we have $U^*_k(f) = f \circ \gamma_k$, we consider the action of the element $\gamma_k = T^k S \in PSL_2(\mathbf{Z})$ on the set $\mathcal{J}_k \times \{s\}$. We have $S(\mathcal{J}_k) = \{x \,|\, \pi(x) = -k - sign(k)[a_2, a_3, \ldots]\}$, and $T^k S(\mathcal{J}_k) = \{x \,|\, \pi(x) = -sign(k)[a_2, a_3, \ldots]\}$. This is the union of all

$$\{x \,|\, \pi(x) = sign(k')[|k'|, a_3, \ldots] \; sign(k') = -sign(k)\}.$$

Thus, we have

$$T^k S : \mathcal{J}_k \times \{s\} \to \cup_{k' : kk' < 0} \; \mathcal{J}_{k'} \times \{s \, S T^{-k}\}, \tag{5.7}$$

we obtain that

$$U^*_k(\chi_{k,s}) \chi_{k',s'} = A_{(k,s),(k',s')} \chi_{k',s'}. \tag{5.8}$$

Thus, we obtain $(S^*_{k,s} S_{k,s}) S_{k',s'} = A_{(k,s),(k',s')} S_{k',s'}$, which is the third Exel–Laca axiom.

For the last axiom, consider the condition that $A(X, Y, b)$ of (5.5) vanishes for all but finitely many $b \in \mathbf{A}$. Given two finite subsets $X, Y \subset \mathbf{Z}^\times \times \mathbf{P}$, the only way that $A(X, Y, b) = 0$ for all but finitely many $b \in \mathbf{Z}^\times \times \mathbf{P}$ is that $A(X, Y, b) = 0$ for all $b$.

In fact, for given $X$ and $Y$, suppose that there exists an element $b = (k, s)$ such that $A(X, Y, b) \neq 0$. This means that, for all $x \in X$ we have $A_{xb} = 1$ and for all $y \in Y$ we have $A_{yb} = 0$. The first condition means that for all $x = (k_x, s_x)$ we have $k_x k < 0$ and $s = s_x\, ST^{k_x}$, while the second condition means that, for all $y = (k_y, s_y)$ we either have $k_y k > 0$ or $k_y k < 0$ and $s \neq s_y\, ST^{k_y}$. Consider then elements of the form $b' = (k', s)$ with $k'k > 0$ and the same $s \in \mathbf{P}$ as $b = (k, s)$. All of these still satisfy $A_{xb'} = 1$ for all $x \in X$ and $A_{yb} = 0$ for all $y \in Y$, since the conditions only depend on the sign of $k$ and on $s \in \mathbf{P}$. Thus, there are infinitely many $b'$ such that $A(X, Y, b') \neq 0$. If $A(X, Y, b) \equiv 0$ for all $b$, the condition of the fourth axiom reduces to

$$\prod_{x \in X} S_x^* S_x \prod_{y \in Y} (1 - S_y^* S_y) = 0. \tag{5.9}$$

Suppose that $A(X, Y, b) \equiv 0$ for all $b \in \mathbf{A}$ but that the expression in (5.9) is non-zero. This means that both $\prod_{x \in X} S_x^* S_x \neq 0$ and $\prod_{y \in Y} (1 - S_y^* S_y) \neq 0$. We first show by induction that, for $X = \{x_1, \ldots, x_N\}$ we can write first product in the form

$$(\prod_{x \in X} S_x^* S_x) P_b = A_{x_1 b} A_{x_2, b} \cdots A_{x_N b} P_b, \tag{5.10}$$

where $P_b = \chi_b = S_b S_b^*$. We know already that this is true for a single point $X = \{x\}$ by (5.8). Suppose it is true for $N$ points. Then we have

$$S_{x_0}^* S_{x_0} (\prod_{i=1}^{N} S_{x_i}^* S_{x_i}) P_b = (\prod_{i=1}^{N} A_{x_i b})\ S_{x_0}^* S_{x_0}\ P_b = A_{x_1 b} A_{x_2, b} \cdots A_{x_N b}\ A_{x_0 b} P_b,$$

which gives the result. Thus, we see that the condition $\prod_{x \in X} S_x^* S_x \neq 0$ implies that, for some $b \in \mathbf{A}$, one has $A_{x,b} \neq 0$ for all $x \in X$, i.e. one has $\prod_{x \in X} A_{x,b} \neq 0$. We analyze similarly the condition $\prod_{y \in Y} (1 - S_y^* S_y) \neq 0$. We show by induction that, for $Y = \{y_1, \ldots, y_M\}$, the product can be written in the form

$$(\prod_{y \in Y} (1 - S_y^* S_y)) P_b = (1 - A_{y_1 b})(1 - A_{y_2 b}) \cdots (1 - A_{y_M b}) P_b. \tag{5.11}$$

First consider the case of a single point $Y = \{y\}$. We have $1 - S_y^* S_y = 1 - U_{k_y}^*(\chi_{k_y, s_y})$ so that using (5.8) we get $(1 - S_y^* S_y) P_b = (1 - A_{yb}) P_b$. We then suppose that the identity (5.11) holds for $M$ points. We obtain, again using (5.8),

$$(1 - S_{y_0}^* S_{y_0}) (\prod_{i=1}^{M} (1 - A_{y_i b})) P_b = (\prod_{i=1}^{M} (1 - A_{y_i b}))\ (1 - A_{y_0 b}) P_b.$$

Thus, the condition $\prod_{y\in Y}(1-S_y^*S_y)\neq 0$ implies that there exists $b\in \mathbf{A}$ such that $\prod_{y\in Y}(1-A_{yb})\neq 0$, which contradicts the fact that we are assuming $A(X,Y,b)\equiv 0$. This proves that the fourth Exel–Laca axiom is satisfied.

To summarize, there are several interesting noncommutative spaces related to the boundary of modular curves $X_\Gamma$: the crossed product algebras $C(\mathbf{P}^1(\mathbf{R})\times\mathbf{P})\rtimes G$ and $C(D_{\mathbf{Q}}\times\mathbf{P})\rtimes G$, and the Exel–Laca algebra $O_A$. This calls for a more detailed investigation of their relations and of the information about the modular tower that each of these noncommutative spaces captures.

**5.7. Limiting modular pseudo–measures.** In [MaMar1] we extended classical modular symbols to include the case of limiting cycles associated to geodesics with irrational endpoints, the limiting modular symbols. The theory of limiting modular symbols was further studied in [Mar] and more recently in [KeS].

We show here that we can similarly define limiting modular pseudo–measures, extending the class of pseudo–measures from finitely additive functions on the Boolean algebra of the $V(e)\subset D_{\mathbf{Q}}$ to a larger class of sets by a limiting procedure.

In the following we assume that the group $W$ where the pseudo–measures take values is also a real (or complex) vector space. If as a vector space $W$ is infinite dimensional, we assume that it is a topological vector space. This is always the case for finite dimension. As before we also assume that $W$ is a left $G$-module.

We recall the following property of the convergents of the continued fraction expansion, cf. [PoWe]. For $\theta\in\mathbf{R}\smallsetminus\mathbf{Q}$, let $q_n$ denote, as before, the successive denominators of the continued fraction expansion. Then the limit

$$\lambda(\theta):=\lim_{n\to\infty}\frac{2\log q_n(\theta)}{n} \tag{5.12}$$

is defined away from an exceptional set $\Omega\subset\mathbf{P}^1(\mathbf{R})$ and where it exists it is equal to the Lyapunov exponent of the Gauss shift.

In [MaMar1] we proved that the limiting modular symbol

$$\{\{*,\theta\}\}=\lim_{t\to\infty}\frac{1}{t}\{x_0,y(t)\}_\Gamma$$

can be computed by the limit

$$\lim_{n\to\infty}\frac{1}{\lambda(\theta)n}\sum_{k=1}^{n+1}\{g_{k-1}(0),g_{k-1}(\infty)\}_\Gamma\in H_1(X_\Gamma,\mathbf{R}) \tag{5.13}$$

where $g_k=g_k(\theta)$ is the matrix in $G$ that implements the action of the $k$–th power of the Gauss shift.

Fix a $W$–valued pseudo–measure $\mu$ on $\mathbf{P}^1(\mathbf{R})$, equivalently thought of as an analytic pseudo–measure on $D_{\mathbf{Q}}$. Consider the class of positively oriented intervals $(\infty,\theta)$ with $\theta \in \mathbf{P}^1(\mathbf{R}) \smallsetminus \mathbf{P}^1(\mathbf{Q})$ and define the limit

$$\mu^{lim}(\infty,\theta) := \lim_{n\to\infty} \frac{1}{\lambda(\theta)n} \sum_{k=1}^{n+1} \mu(g_{k-1}(\infty), g_{k-1}(0)) \in W. \tag{5.14}$$

This is defined away from the exceptional set $\Omega$ in $\mathbf{P}^1(\mathbf{R})$ which contains $\mathbf{P}^1(\mathbf{Q})$ as well as the irrational points where either the limit defining $\lambda(\theta)$ does not exist or the limit of the

$$\frac{1}{n}\sum_{k=1}^{n+1} \mu(g_{k-1}(\infty), g_{k-1}(0))$$

does not exist.

Similarly, for $\theta \in \mathbf{P}^1(\mathbf{R}) \smallsetminus \Omega$ we set

$$\mu^{lim}(\theta,\infty) = \lim_{n\to\infty} \frac{1}{\lambda(\theta)n} \sum_{k=1}^{n+1} \mu(g_{k-1}(0,\infty)). \tag{5.15}$$

We then set

$$\mu^{lim}(\theta,\eta) := \mu^{lim}(\theta,\infty) + \mu^{lim}(\infty,\eta). \tag{5.16}$$

**5.7.1. Lemma.** *The function $\mu^{lim}$ defined as above satisfies*

$$\mu^{lim}(\eta,\theta) = -\mu^{lim}(\theta,\eta) \qquad \mu^{lim}(\theta,\eta) + \mu^{lim}(\theta,\zeta) + \mu^{lim}(\zeta,\theta) = 0$$

*for all $\theta,\eta,\zeta \in \mathbf{P}^1(\mathbf{R}) \smallsetminus \Omega$.*

**Proof.** Since $(\beta,\alpha) = g(0,\infty) = g\sigma(\infty,0)$, we have $\mu(\beta_k,\alpha_k) = \mu(g_k(0,\infty)) = \mu(g_k\sigma(\infty,0)) = -\mu(\alpha_k,\beta_k)$ so that

$$\mu^{lim}(\theta,\infty) = -\mu^{lim}(\infty,\theta).$$

We then have $\mu^{lim}(\eta,\theta) = \mu^{lim}(\infty,\theta) + \mu^{lim}(\eta,\infty) = -\mu^{lim}(\theta,\eta)$. The argument for the second identity is similar.

Thus, the limiting pseudo–measure $\mu^{lim}$ defines a finitely additive function on the Boolean algebra generated by the intervals $(\theta,\eta)$ with $\theta,\eta \in \mathbf{P}^1(\mathbf{R}) \smallsetminus \Omega$. The limiting pseudo–measure $\mu^{lim}$ is $G$–modular if $\mu$ is $G$–modular.

**5.7.2. Limiting pseudo–measures and currents.** In terms of currents on the tree $\mathcal{T}$, we can describe the limit computing $\mu^{lim}$ as a process of averaging the current $\mathbf{c}$ over edges along a path.

**5.7.3. Lemma.** *Let $\mu$ be a $W$–valued pseudo–measure and $\mathbf{c}$ the corresponding current on $\mathcal{T}$. The limiting pseudo–measure $\mu^{lim}$ is computed by the limit*

$$\mu^{lim}(\infty,\theta) = \lim_{n\to\infty} \frac{1}{\lambda(\theta)n} \sum_{k=1}^{n} \mathbf{c}(e_k).$$

**Proof.** An irrational point $\theta$ in $\partial\mathcal{T}$ corresponds to a unique admissible infinite path in the tree $\mathcal{T}$ starting from a chosen base vertex. We describe such a path as an infinite admissible sequence of oriented edges $e_1 e_2 \cdots e_n \cdots$. To each such edge there corresponds an open set $V(e_k)$ in $\partial\mathcal{T}$ with the property that $\cap_k V(e_k) = \{\theta\}$. These correspond under the map $\Upsilon : \partial\mathcal{T} \to \mathbf{P}^1(\mathbf{R})$ to intervals $[g_k(\infty), g_k(0)]$. Thus, the expression computing the limiting pseudo–measure can be written equivalently via the limit of the averages

$$\frac{1}{n} \sum_{k=1}^{n} \mathbf{c}(e_k).$$

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